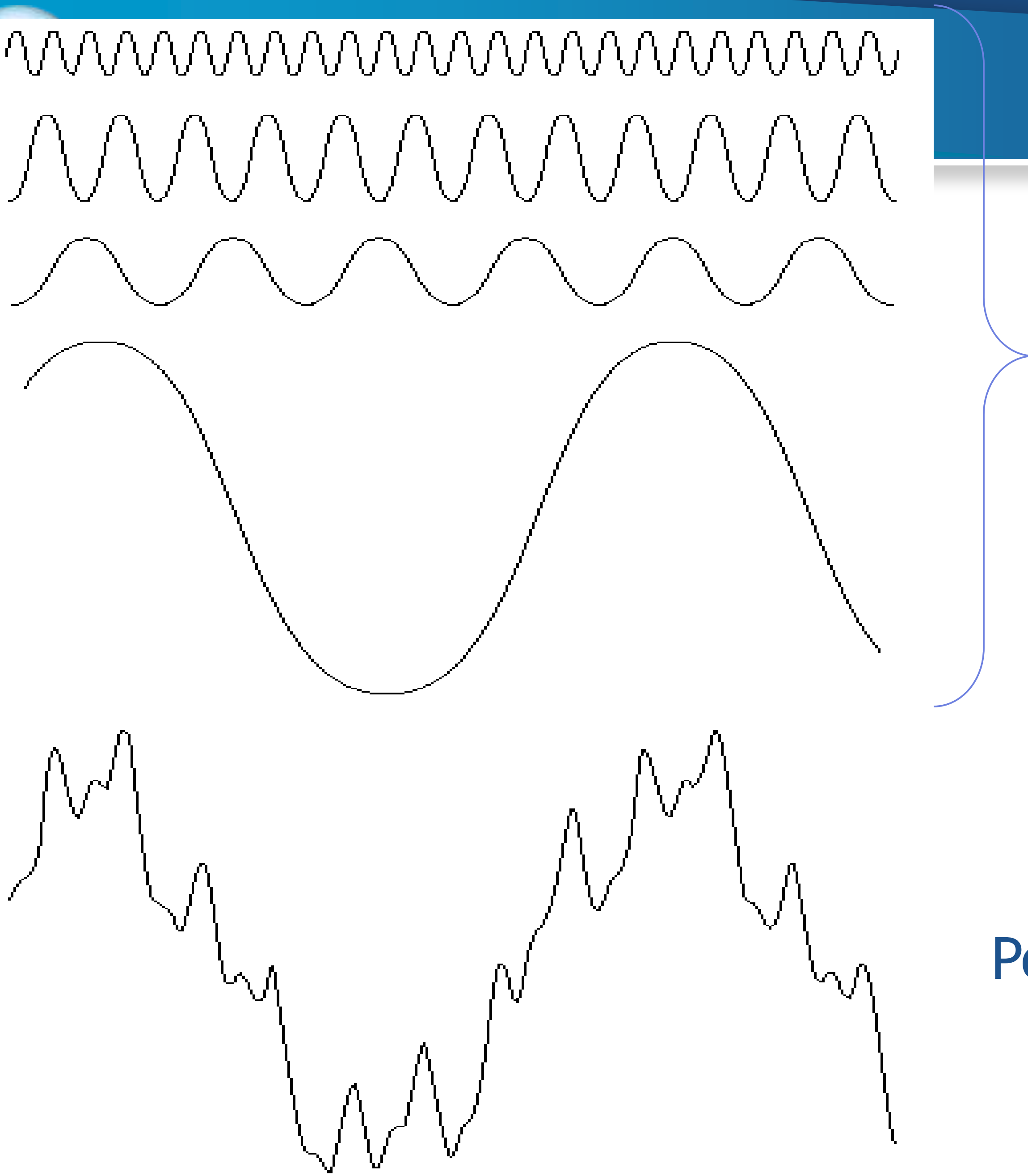
Chapter 4
Image Enhancement in the Frequency Domain

4.1 Background

- 1807, French math. *Fourier*
  - Any function that periodically repeats itself can be expressed as the sum of **sines** and/or **cosines** of different frequencies, each multiplied by a different coefficient (*Fourier series*)
- 2-D transform can be applied to image enhancement, restoration, encoding, and description.
- Fourier transform (FT)
  - Fourier's idea – fig. 4.1.



Frequency	Weight
f_1	w_1
f_2	w_2
f_3	w_3
f_4	w_4

||

Periodic function f

4.2 Introduction to the Fourier transform

◎ Let f(x) be a continuous function of a real variable x. The Fourier transform of f(x), denoted as $\Im\{f(x)\}$, is defined by

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx$$

where $j = \sqrt{-1}$

◎ Given F(u), f(x) can be obtained by using inverse Fourier transform

$$\Im^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

◎ f(x) is real, F(u) is complex

4.2 Introduction to the Fourier transform

◎ $F(u) = R(u) + jI(u)$

$= |F(u)|\, e^{j\phi(u)}$

◎ $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$ ※ phase angle ※

◎ $|F(u)| = \left|R^2(u) + I^2(u)\right|^{1/2}$ ※ Fourier spectrum of f(x) ※

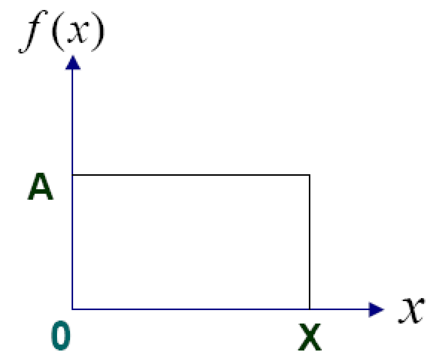
◎ $P(u) = |F(u)|^2$ ※ power spectrum of f(x) ※

◎ u is called the frequency variable

◎ Fig. 3.1 shows a simple function and its Fourier spectrum

◎ Let f(x,y) be a continuous function of two real variables x and y.

Example 1



$$F(u)=\int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]\, dx$$

$$=\int_{0}^{X} A\exp[-j2\pi ux]\, dx$$

$$=\frac{-A}{j2\pi u}\left[\exp(-j2\pi ux)\right]_0^X=\frac{-A}{j2\pi u}\left[\exp(-j2\pi uX)-1\right]$$

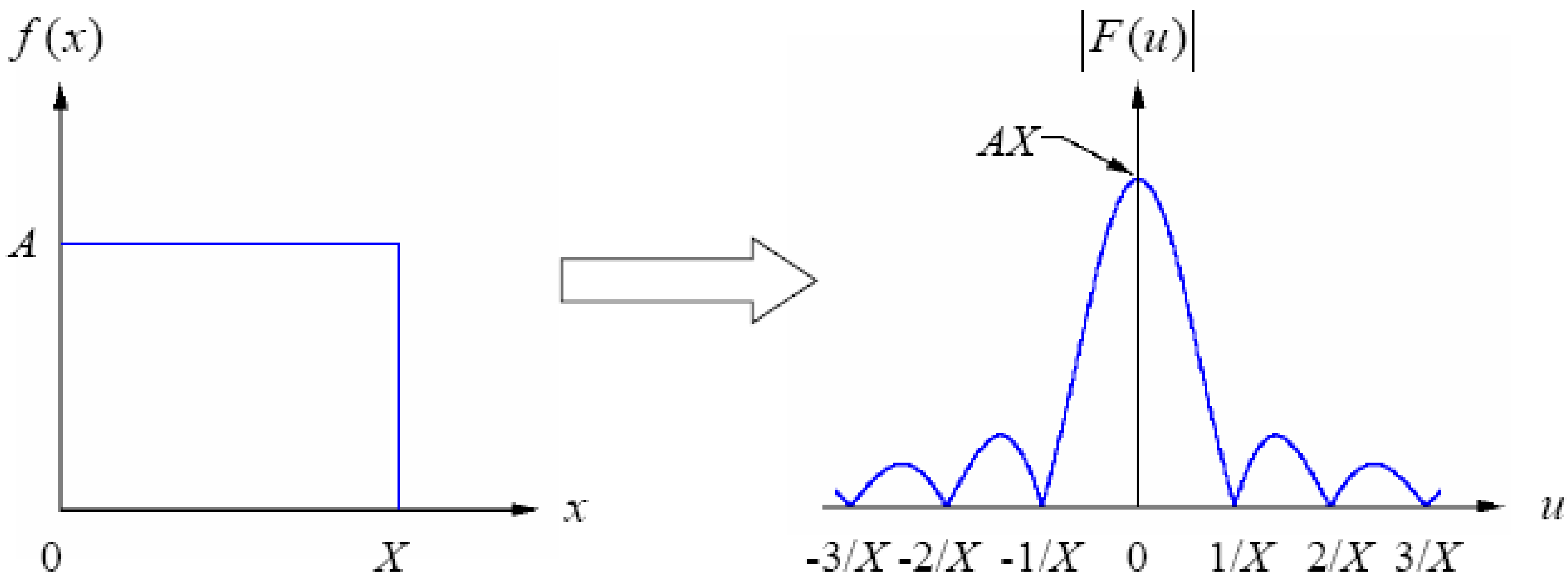
$$=\frac{A}{j2\pi u}\left[\exp(j\pi uX)-\exp(-j\pi uX)\right]\exp(-j\pi uX)$$

$$=\frac{A}{\pi u}\sin(\pi uX)\exp(-j\pi uX)$$

$$|F(u)|=\left|\frac{A}{\pi u}\sin(\pi uX)\right|\left|\exp(-j\pi uX)\right|=AX\left|\frac{\sin\pi uX}{\pi uX}\right|$$

$$=AX\left|\text{sinc }\pi uX\right|$$

✓Example:



4.2 Introduction to the Fourier transform

2-D Fourier transform of f(x,y), denoted as $\Im\{f(x,y)\}$, is defined by

$$\Im\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy$$

◎ Given F(u,v), f(x,y) can be obtained by using inverse Fourier transform

$$\Im^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[j2\pi(ux+vy)]dudv$$

4.2 Introduction to the Fourier transform

◎ $F(u,v) = R(u,v) + jI(u,v)$

$= |F(u,v)|\, e^{j\phi(u,v)}$

◎ $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$ ※ phase angle ※

◎ $|F(u,v)| = \left|R^2(u,v) + I^2(u,v)\right|^{1/2}$ ※ Fourier spectrum of f(x,y)

◎ $P(u,v) = |F(u,v)|^2$ ※ power spectrum of f(x,y) ※

◎ u and v are called the frequency variables

Example 4.1. Fourier spectra of two simple 1-D functions. (Fig. 4.2)

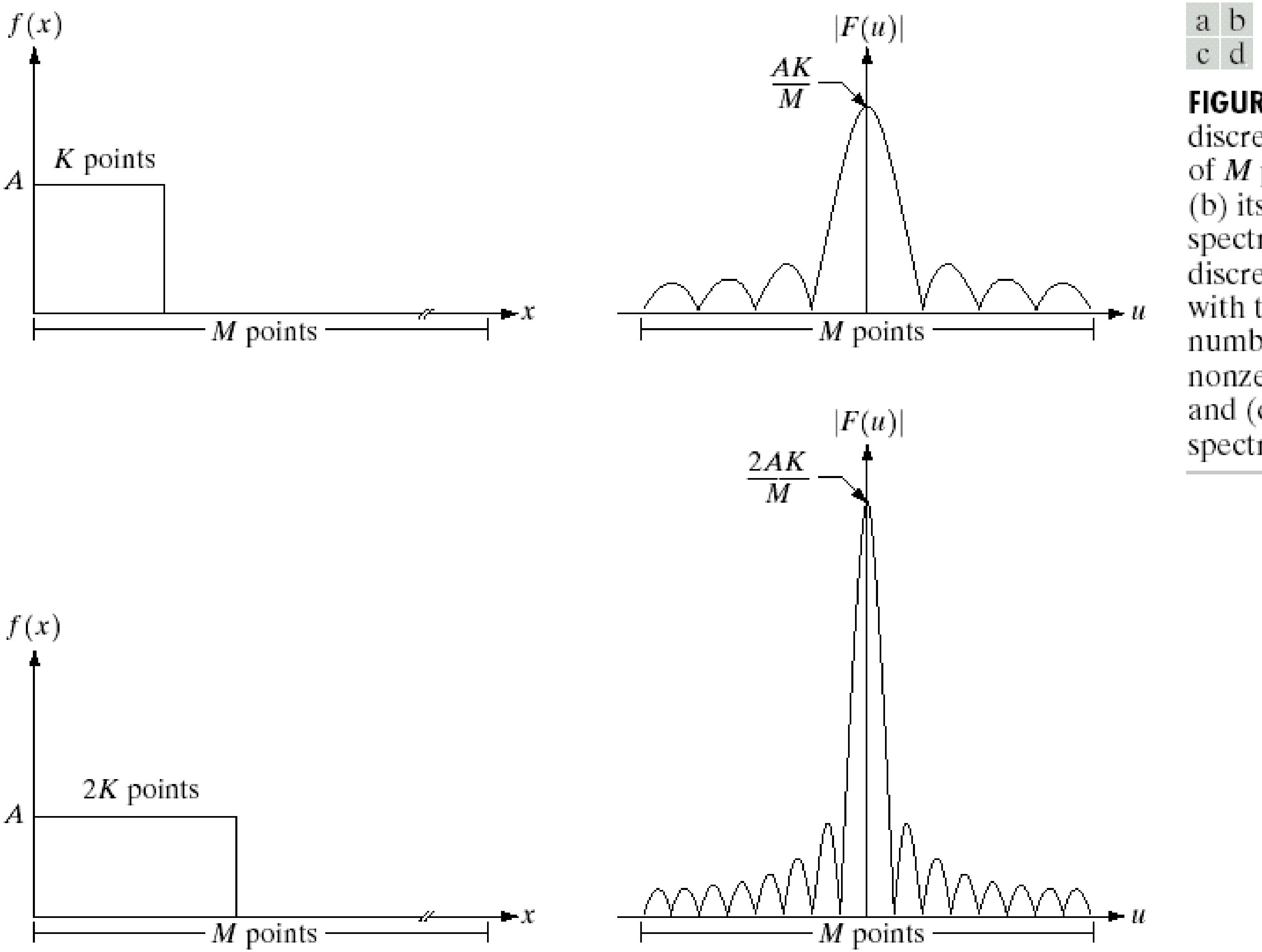
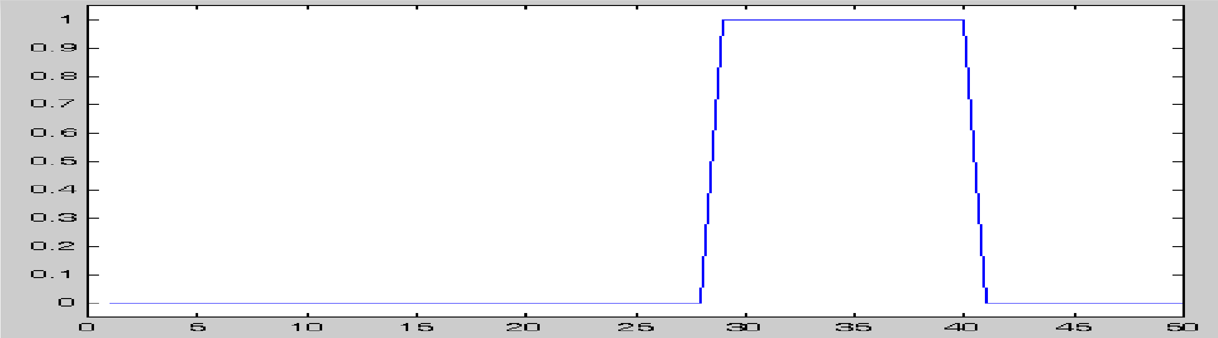
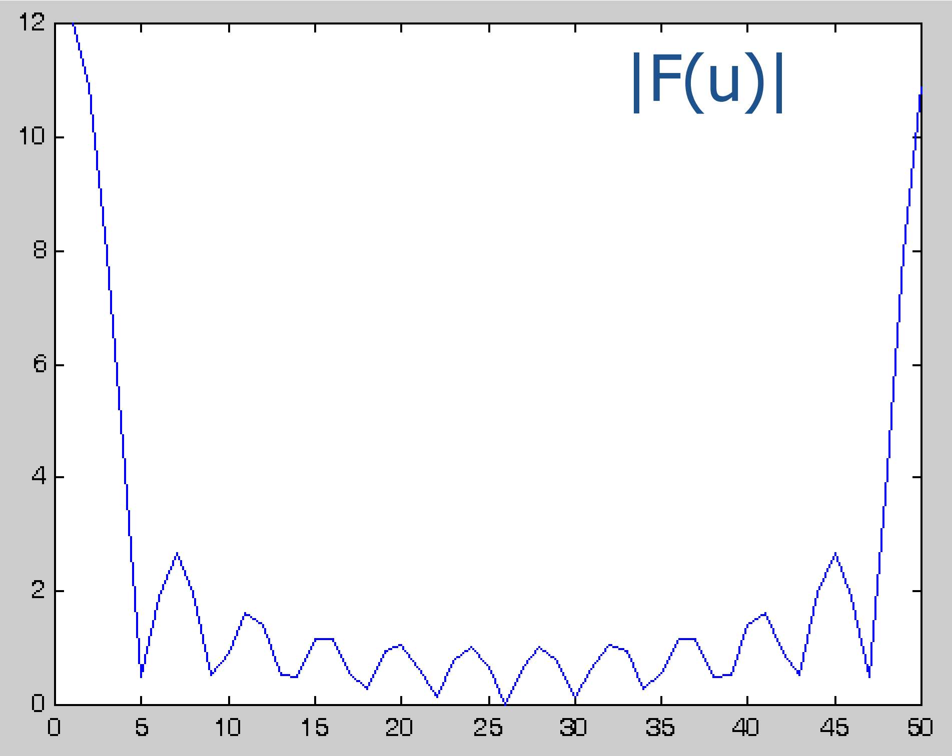
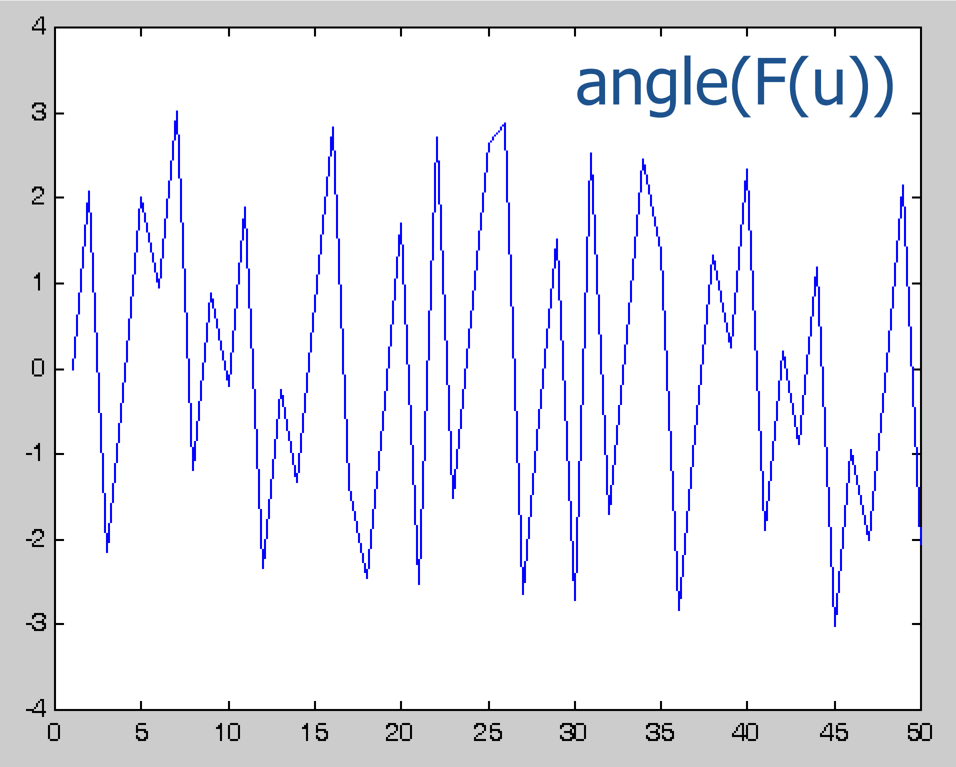


FIGURE 4.2 (a) A discrete function of *M* points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

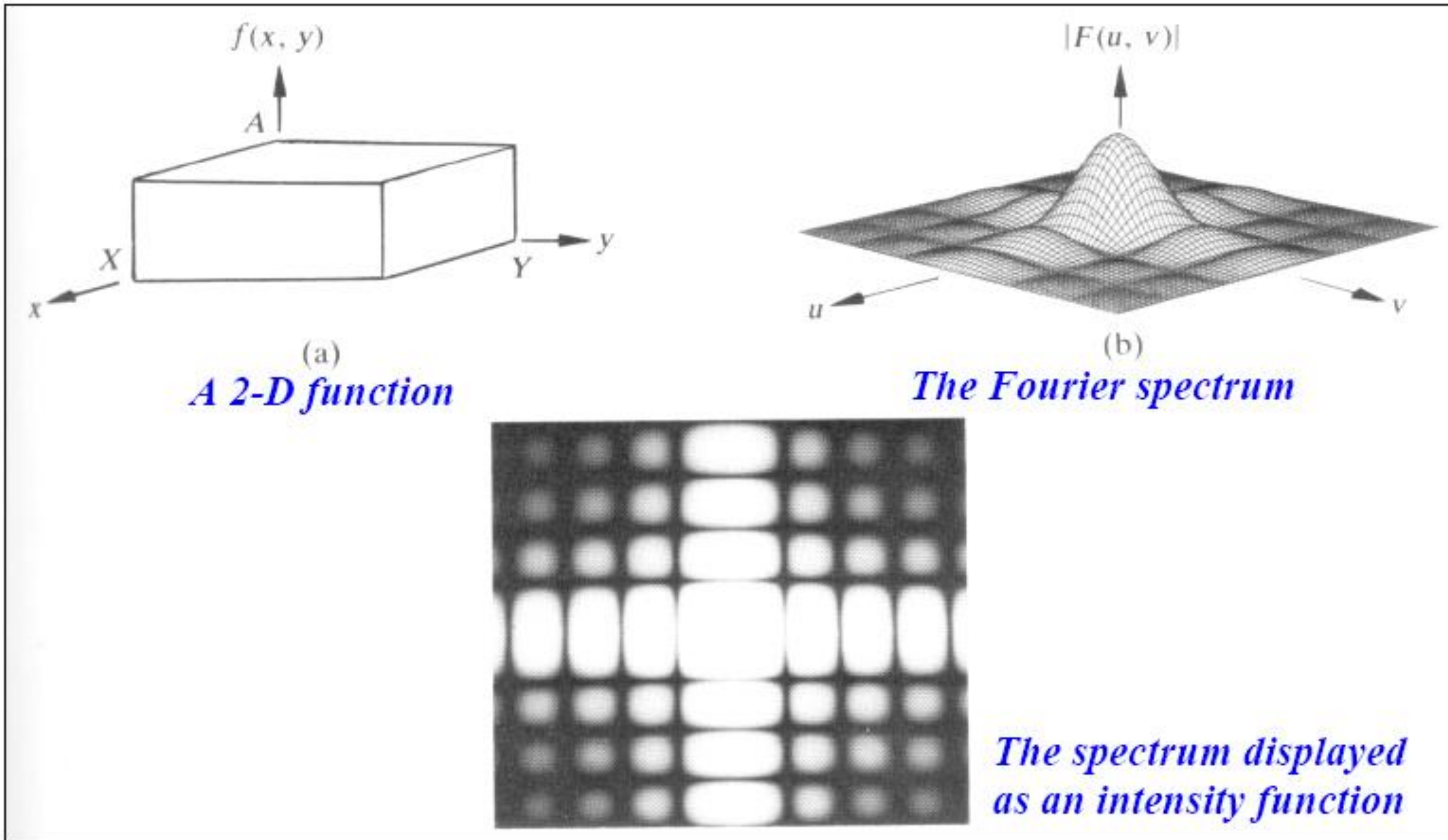
DFT example for real signal







4.2 Introduction to the Fourier transform



A 2-D function

The Fourier spectrum

The spectrum displayed as an intensity function

4.2 Introduction to the Fourier transform

- The discrete Fourier transform (DFT)
  - For 1-D transform: Let the sequence{f(0),f(1),...,f(N-1)} be n real points, the discrete Fourier transform pair is given by

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \exp[-j2\pi ux / N]$$

for u=0,1,2,...,N-1, and

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp[j2\pi ux / N]$$

for x=0,1,2,...,N-1

$$\boxed{DFT \quad Exp.}$$

$$F(0) = \sum_{x=0}^{3} f(x)\exp[0]$$

$$= [f(0)+f(1)+f(2)+f(3)]$$

$$=(2+3+4+4)=13$$

$$F(1) = \sum f(x)\exp[-j2\pi x/4]$$

$$=2e^{0} + 3e^{-j\frac{\pi}{2}} + 4e^{-j\pi} + 4e^{-j\frac{3\pi}{2}}$$

$$=2-3j-4+4j$$

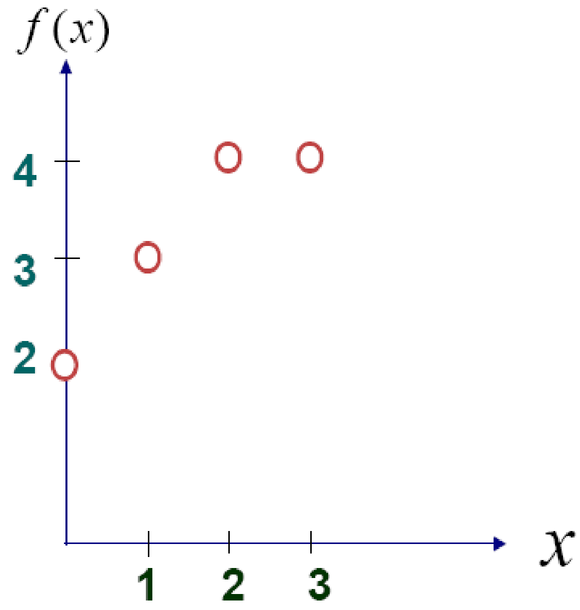
$$=-2+j$$

$$F(2) = \sum f(x)\exp[-j4\pi x/4]$$

$$= 2e^{0} + 3e^{-j\pi} + 4e^{-j2\pi} + 4e^{-j3\pi}$$

$$= 2-3+4-4=-1$$

$$F(3) = -2-j$$



4.2 Introduction to the Fourier transform

– For 2-D transform: in the two-variable case the discrete Fourier transform pair is

$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux+vy)/N]$$

for u,v = 0,1,2,...,N-1, and

$$f(x,y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) \exp[j2\pi(ux+vy)/N]$$

for x,y = 0,1,...,N-1.

4.2 Introduction to the Fourier transform

(1) A continuous function *f(x,y)* is discretized into a sequence

$$\{f(x_0,y_0),\ f(x_0+\Delta x,y_0),\ f(x_0+\Delta x,y_0+\Delta y),\ldots,\ f(x_0+[M-1]\Delta x,y_0+[N-1]\Delta y)\}$$

(2) Define

$$g(x,y)=f(x_0+x\Delta x,y_0+y\Delta y)\quad x=1,\ldots,M\text{-}1,\quad y=1,\ldots,N\text{-}1$$

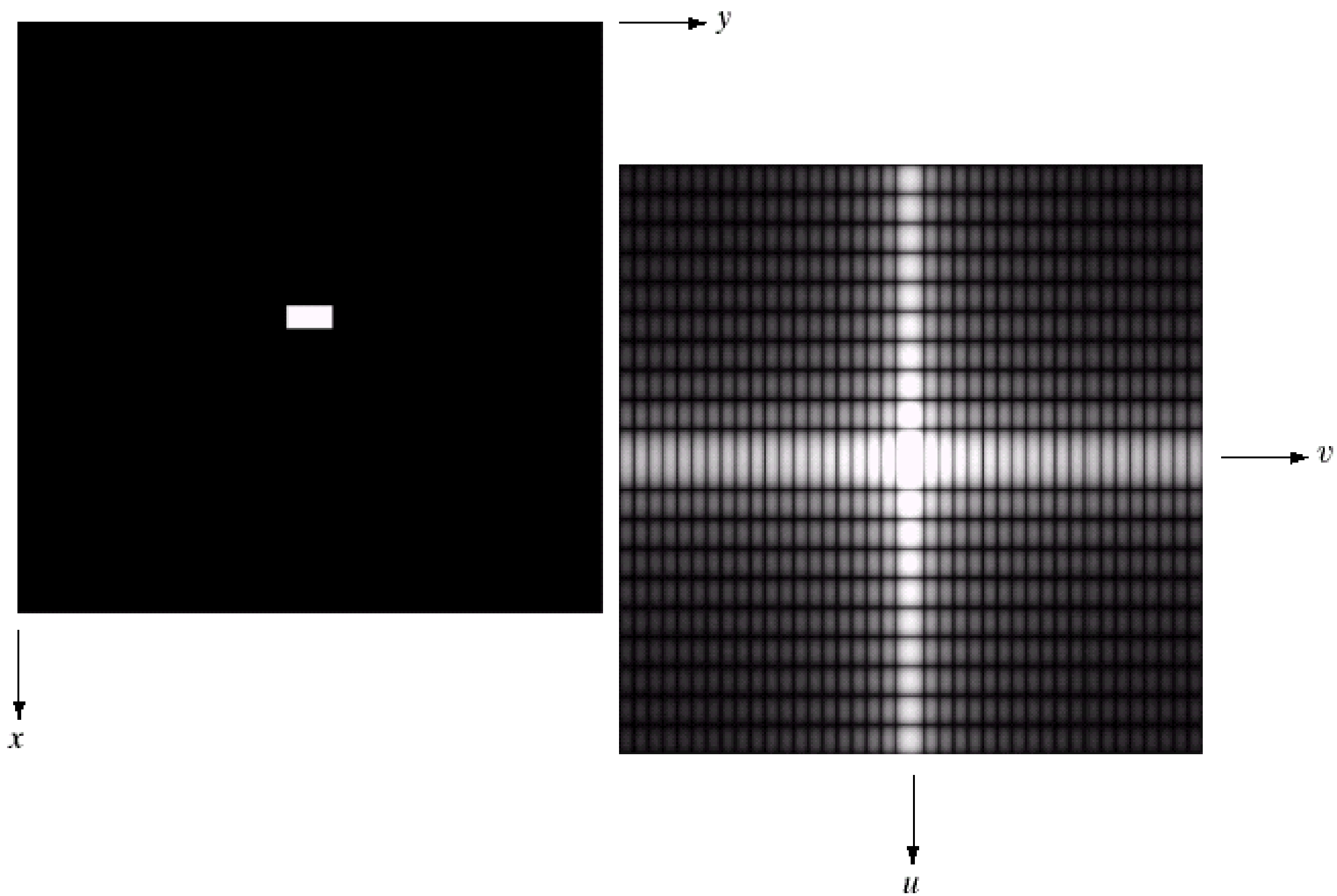
(3) The discrete Fourier transform G(u,v) of *g(x,y)* satisfies

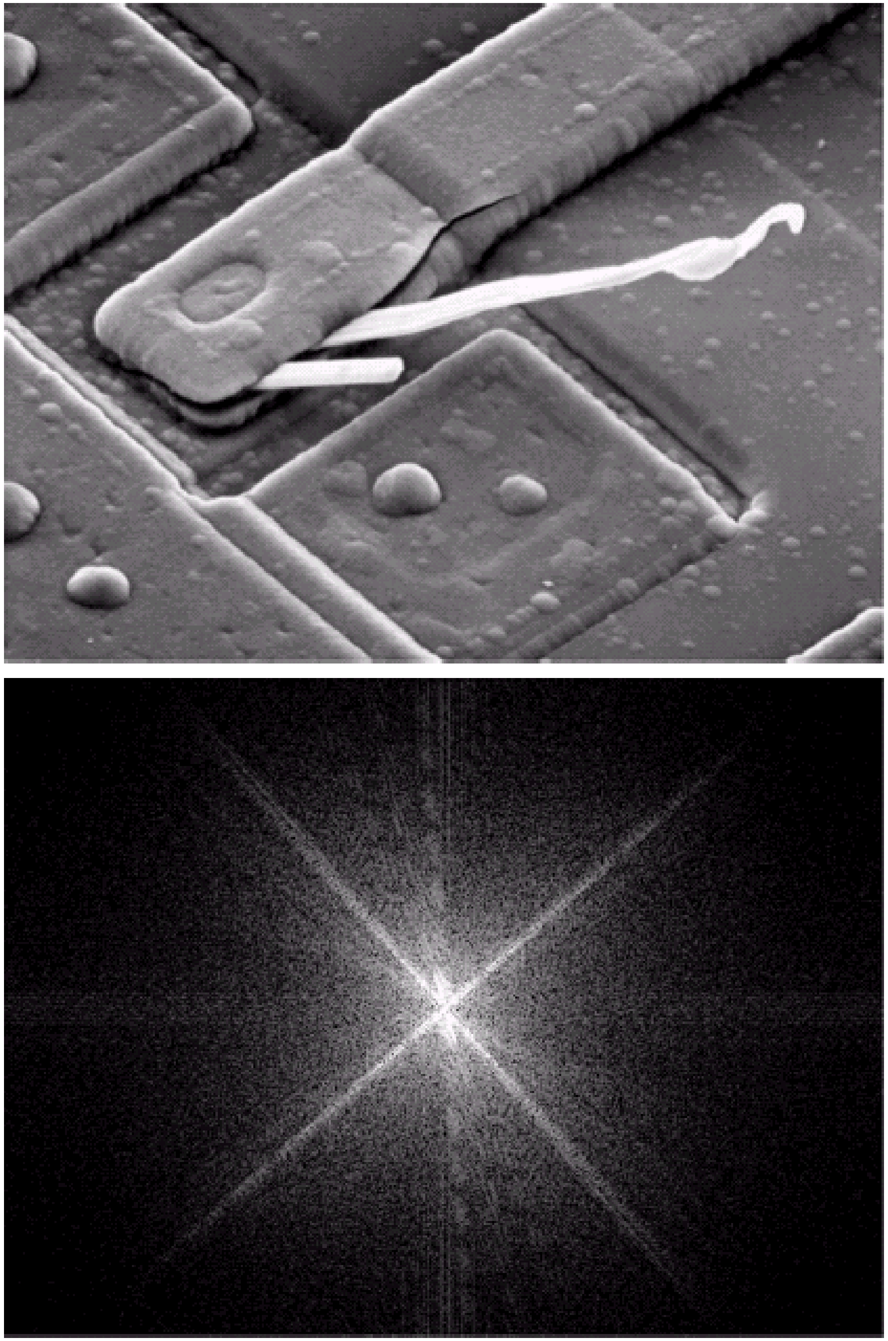
$$G(u,v) = F(u\Delta u, v\Delta v) \quad \text{and} \quad \Delta u=\frac{1}{M\Delta x},\ \Delta v=\frac{1}{M\Delta y}$$

Example 4.2. Centered spectrum of a simple 2-D functions. (Fig. 4.3)

Example 4.3. Fourier spectrum (Fig. 4.4) a b

FIGURE 4.3
(a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels.
(b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



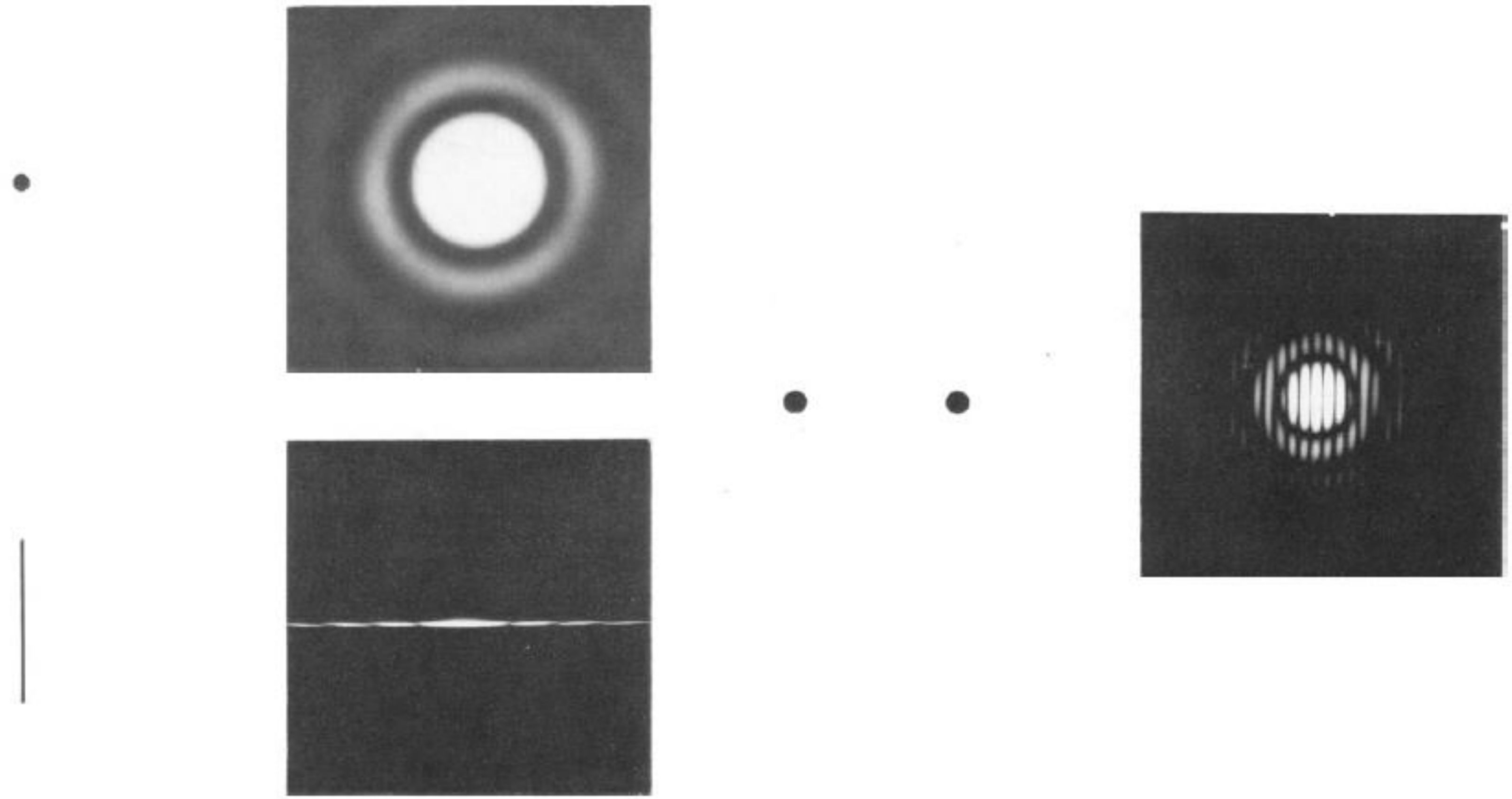


a
b

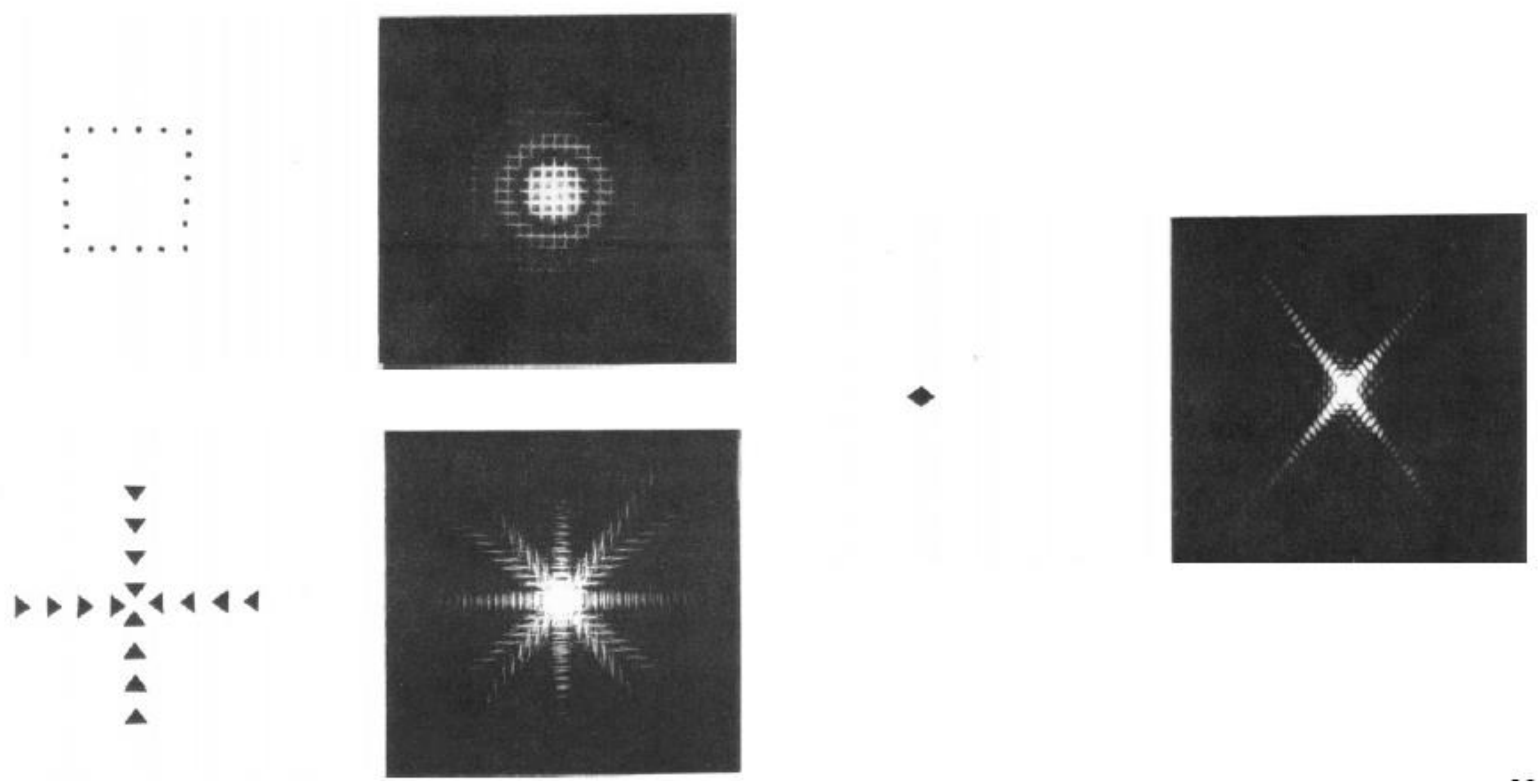
FIGURE 4.4
(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

4.2 Introduction to the Fourier transform

✓ Example:



✓Example:



Filtering in the Frequency Domain

$$G(u,v) = H(u,v)F(u,v)$$

$$F(u,v) = \Im\left[f(x,y)\right]$$

$$g(x,y) = \Im^{-1}\left[G(u,v)\right]$$

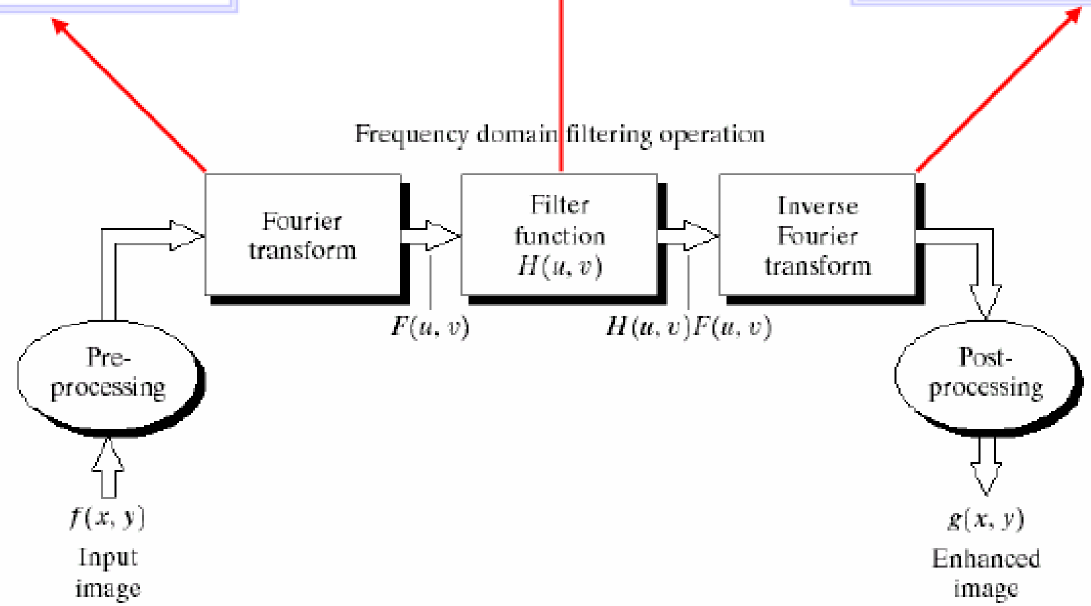
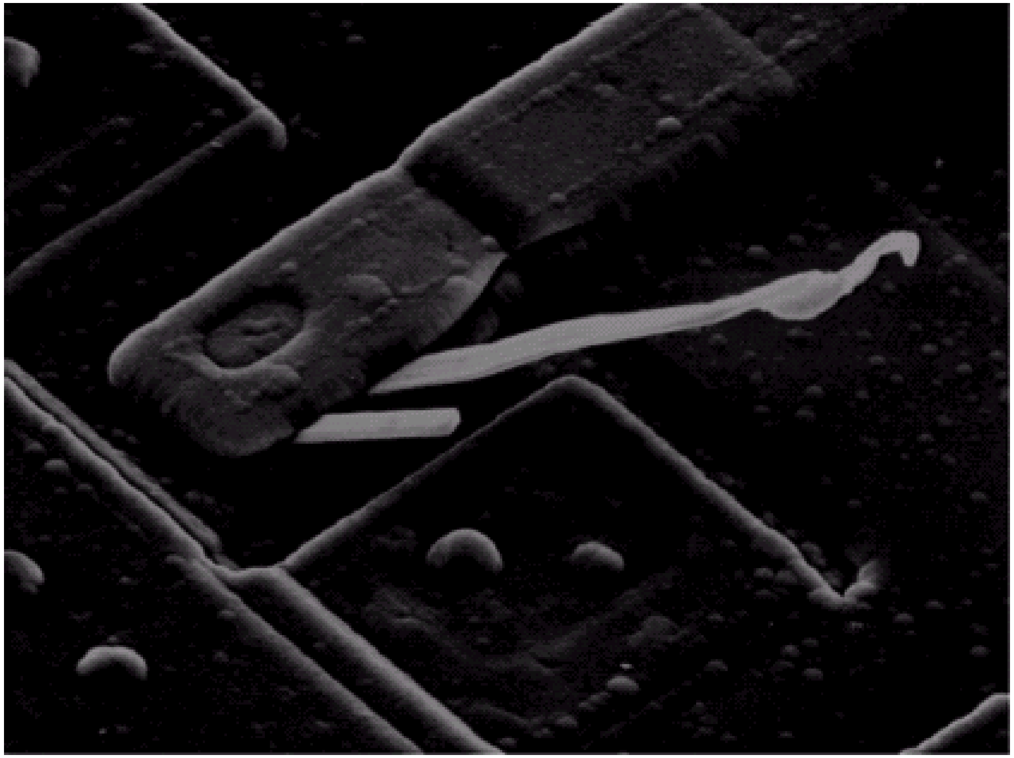


FIGURE 4.5 Basic steps for filtering in the frequency domain.

FIGURE 4.6
Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.



Relation between average value of a function and its Fourier transform:

$$\bar{f}(x,y) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y) W_N^{ux} W_N^{vy}$$

$$\Rightarrow \bar{f}(x,y) = \frac{1}{N} F(0,0)$$

Connection between spatial and frequency filters

- Convolution theorem

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

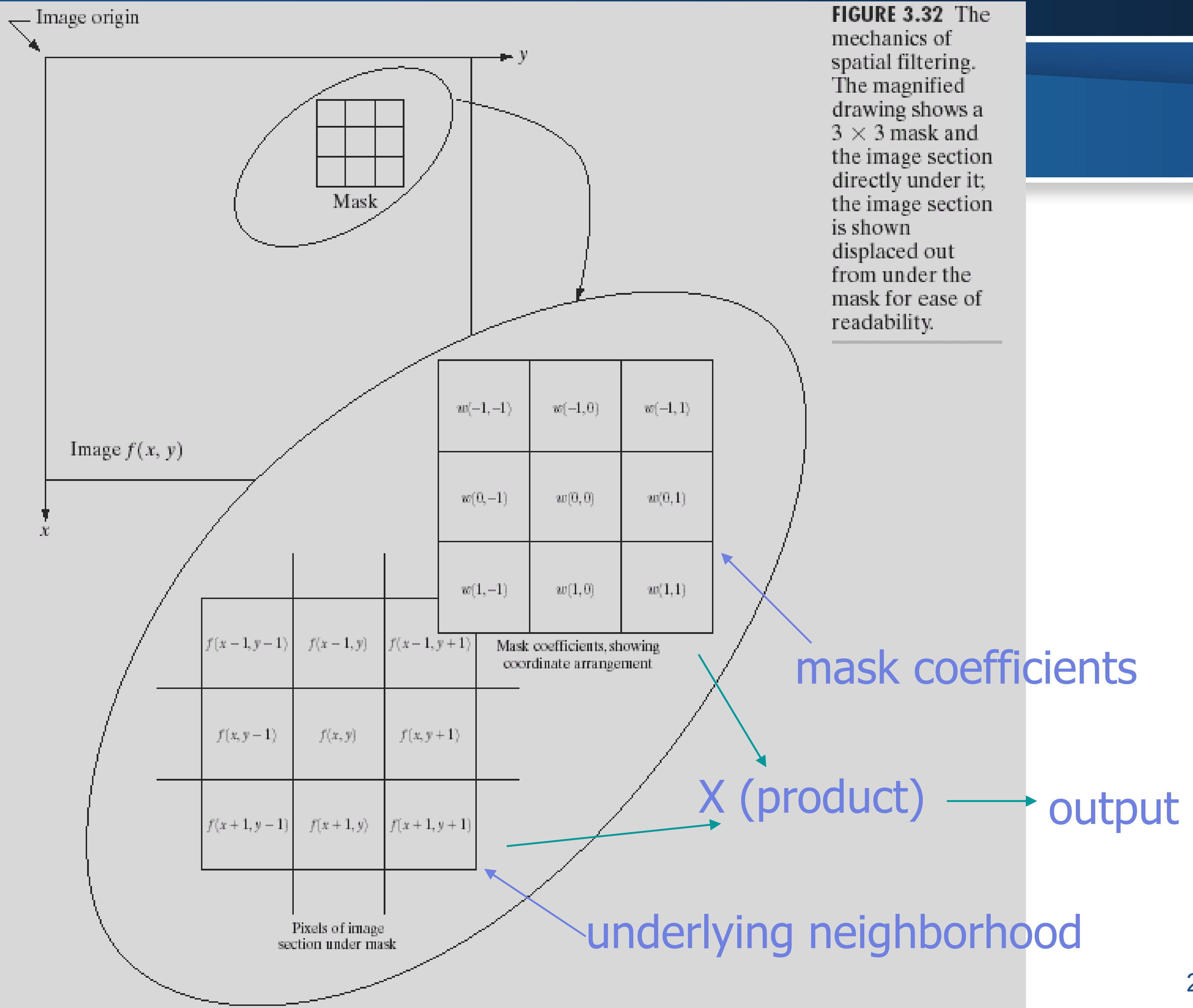
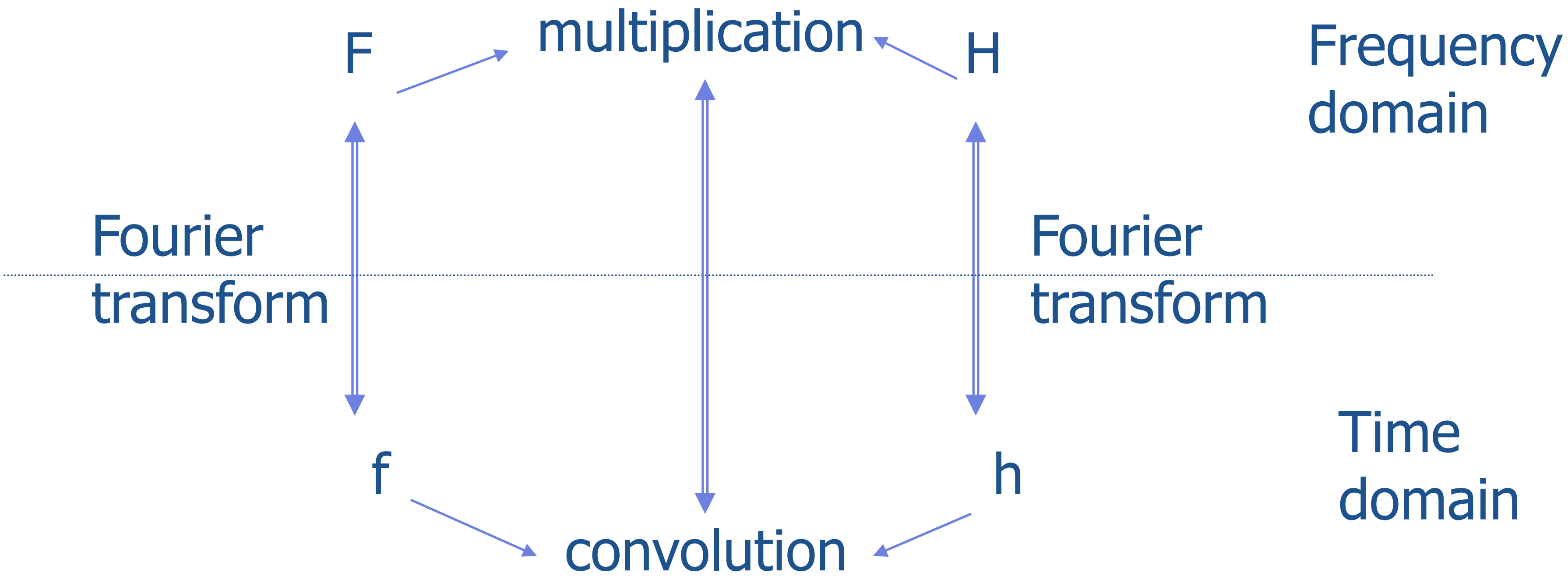


FIGURE 3.32 The mechanics of spatial filtering. The magnified drawing shows a 3 × 3 mask and the image section directly under it; the image section is shown displaced out from under the mask for ease of readability.

Convolution theorem

$$f(x,y)*h(x,y) \Leftrightarrow F(u,v)H(u,v)$$



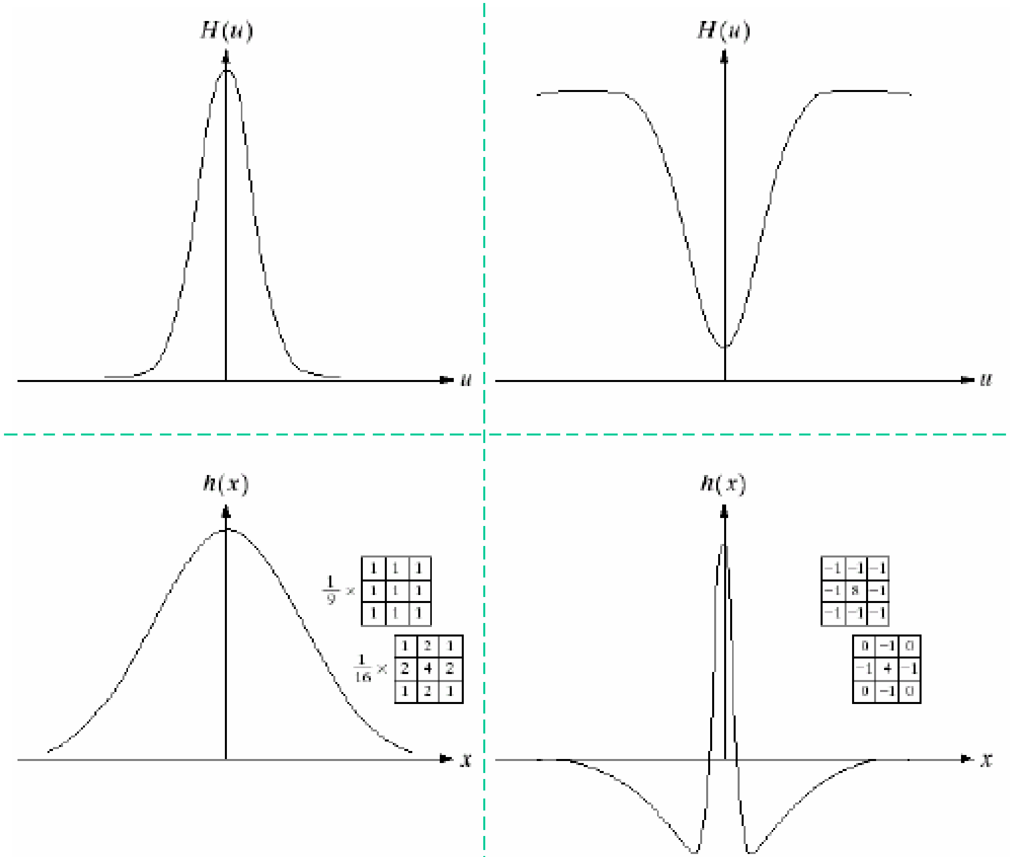
Convolution Theorem

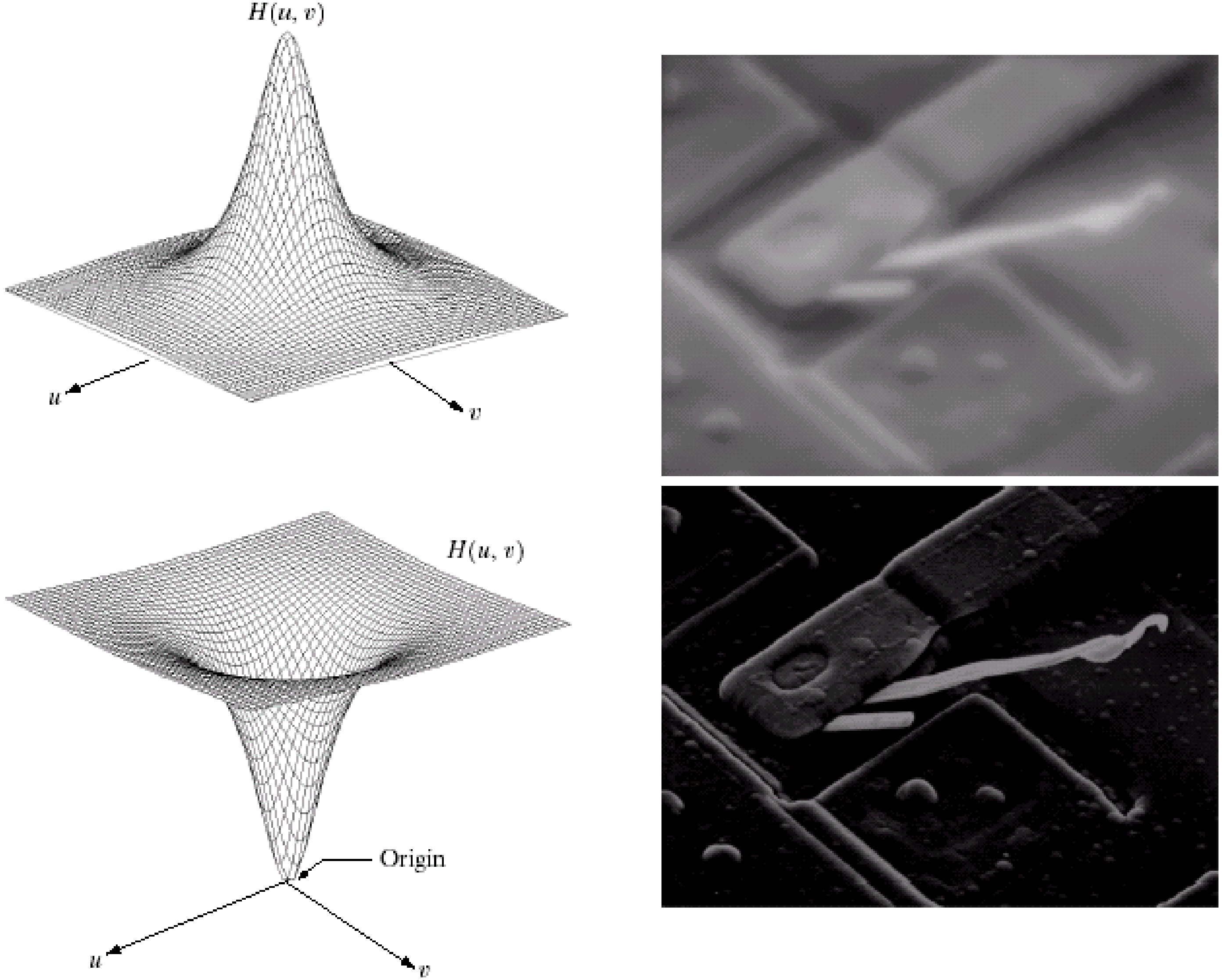
$$f(x,y)*h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

Lowpass | Highpass

Frequency

Spatial

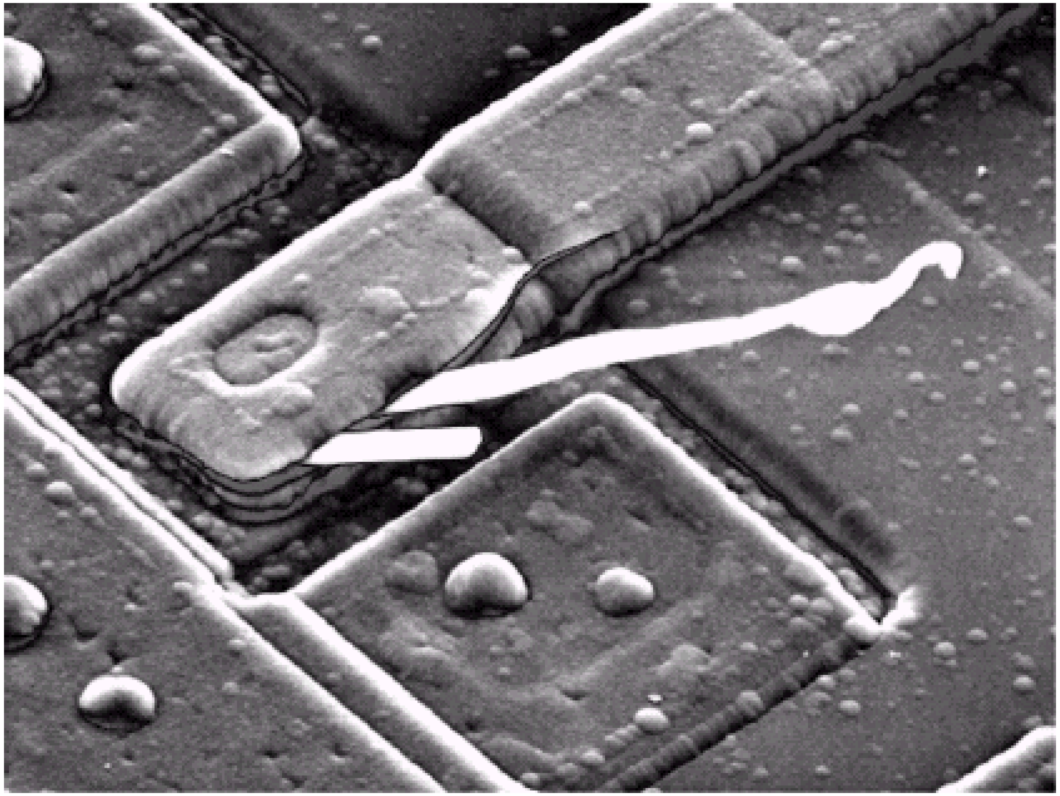


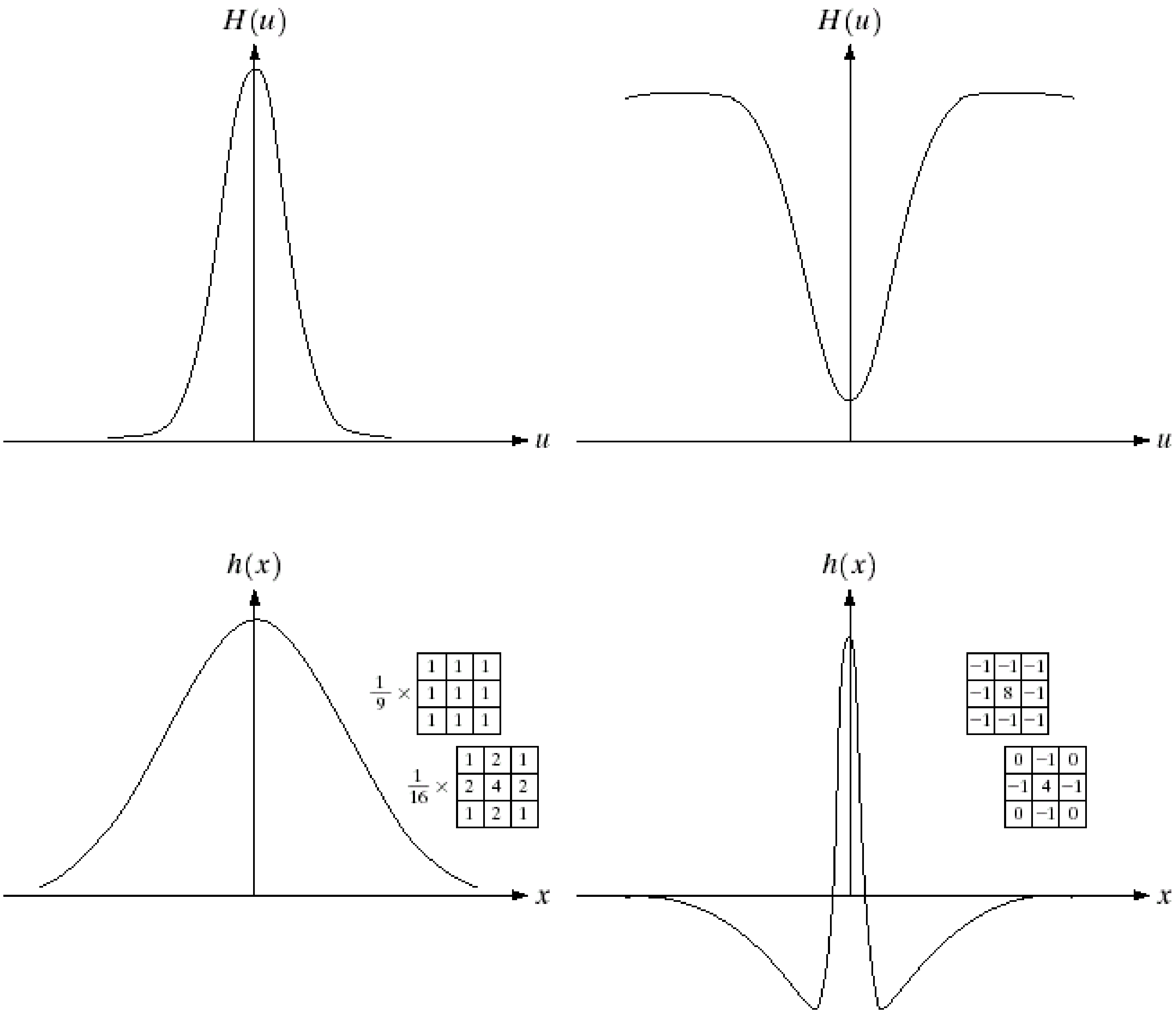


a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

FIGURE 4.8
Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).





a b
c d

FIGURE 4.9
(a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

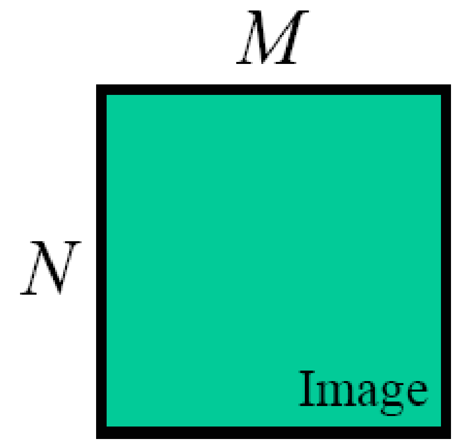
4.3 Smoothing Frequency-Domain Filters

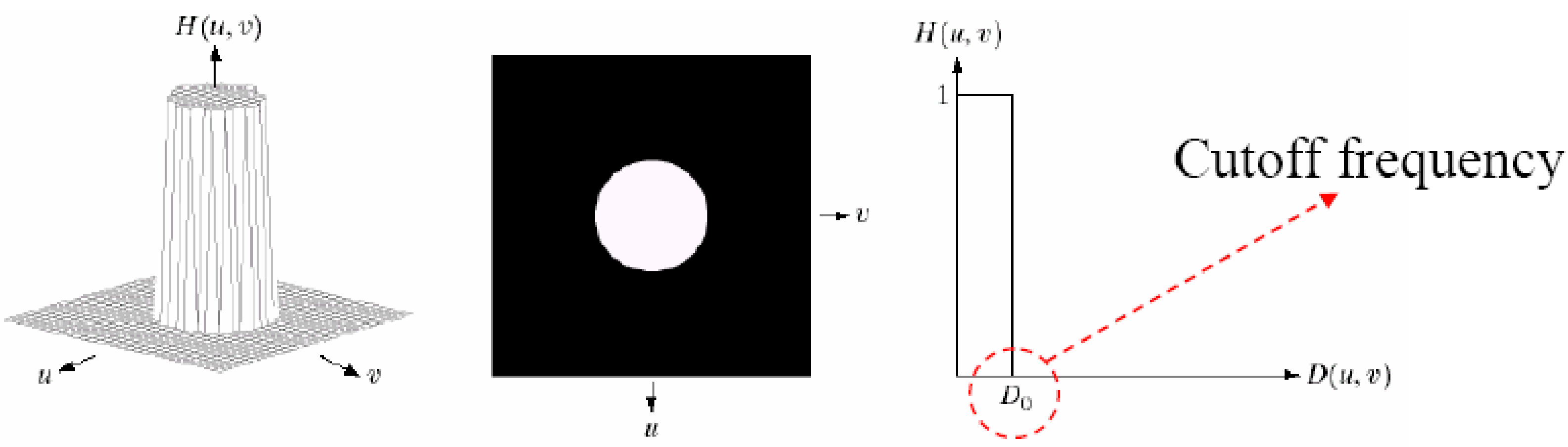
- Ideal lowpass filters (ILPF)
  - Fig. 4.10
  - Example 4.4. Image power as a function of distance from the origin of the DFT. (Fig. 4.11) (Fig. 4.12)

Ideal lowpass filters

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)> D_0\end{cases}$$

$$D(u,v)=\left[\left(u-\tfrac{M}{2}\right)^2+\left(v-\tfrac{N}{2}\right)^2\right]^{1/2}$$





a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal lowpass filters

Total image power : $P_T = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} P(u,v)$

Enclosed power : $\alpha = 100\left[\frac{\sum_u \sum_v P(u,v)}{P_T}\right]$

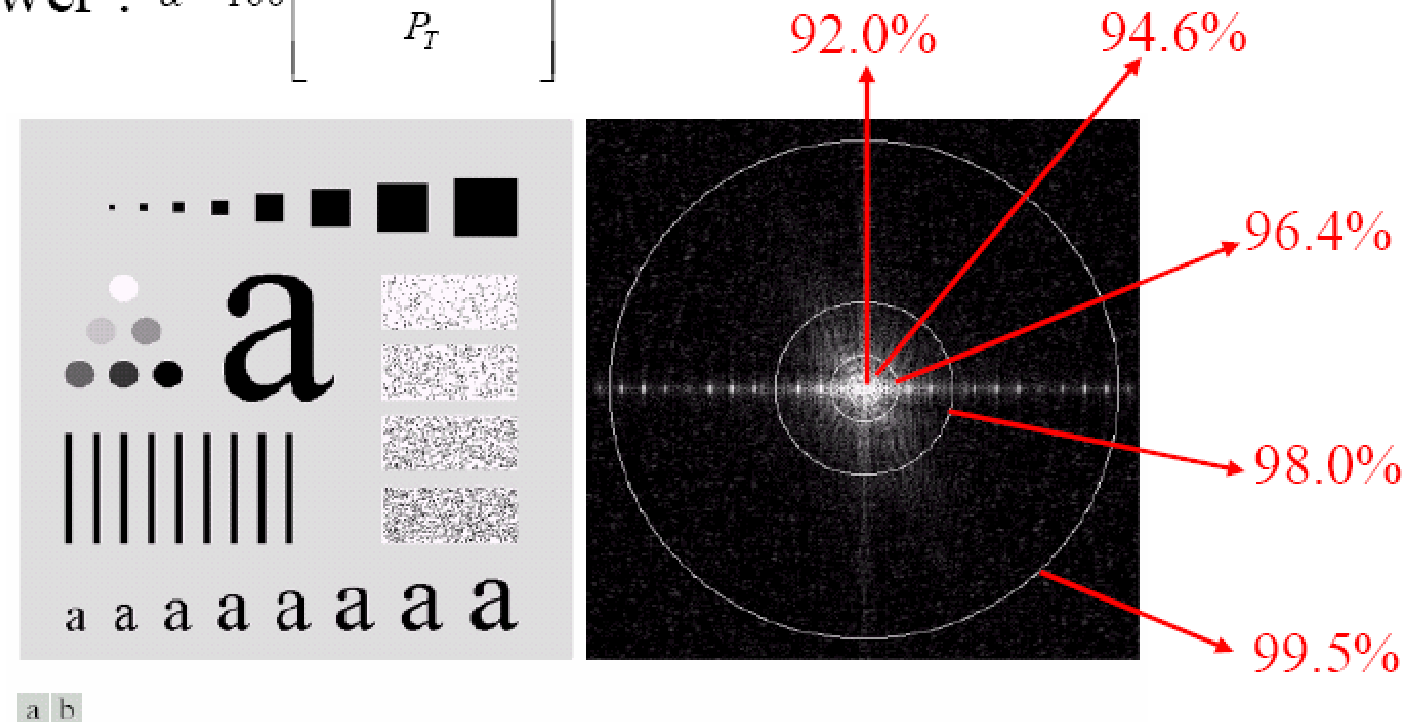
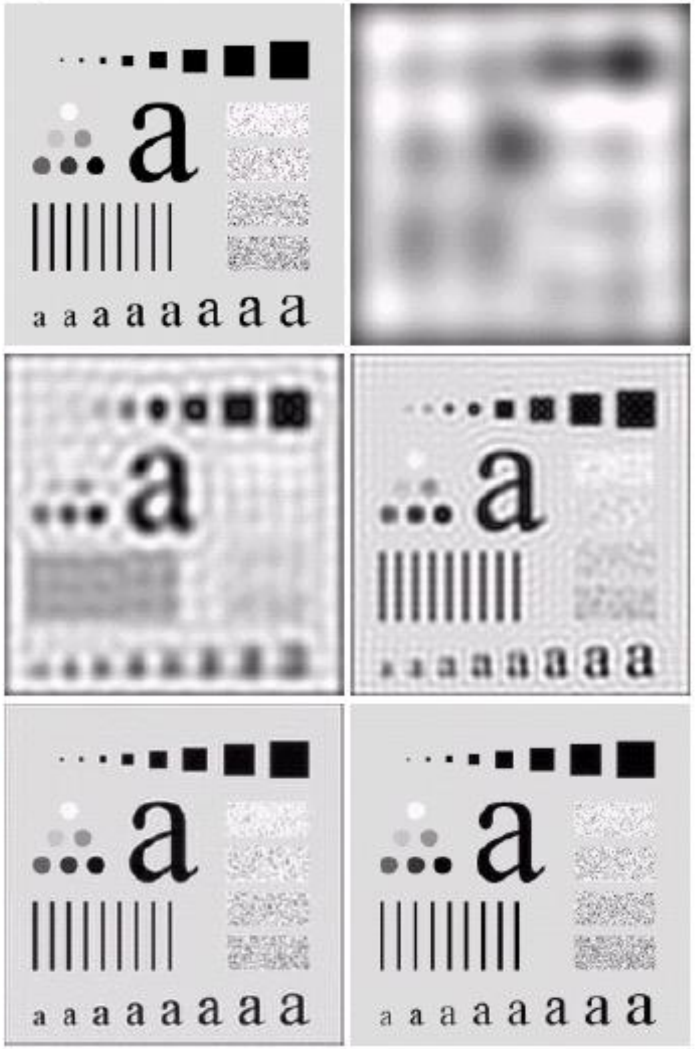


FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Lost power



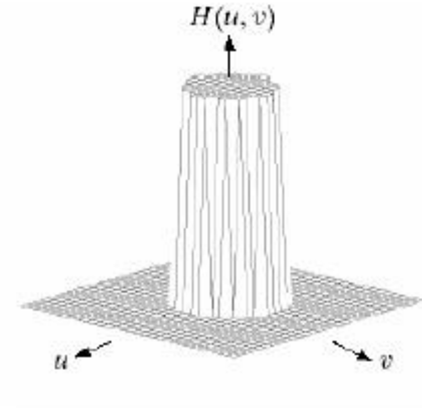
8%

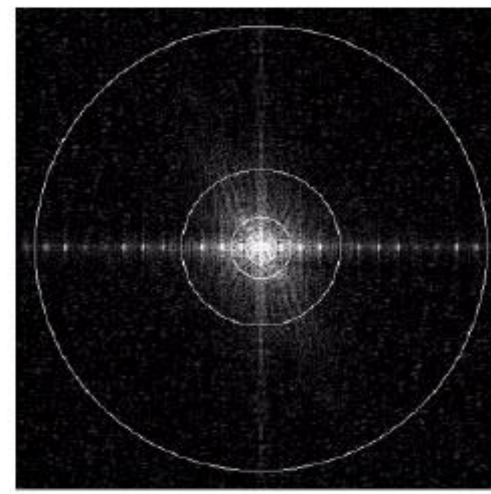
5.4%

3.6%

2%

0.5%





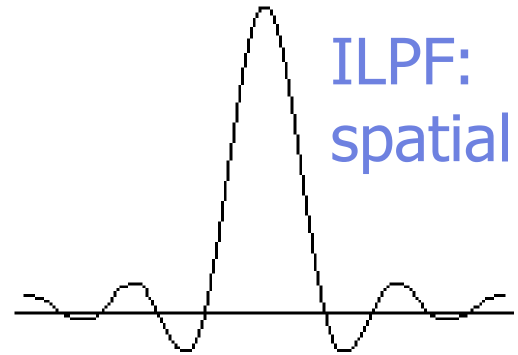
a b
c d
e f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

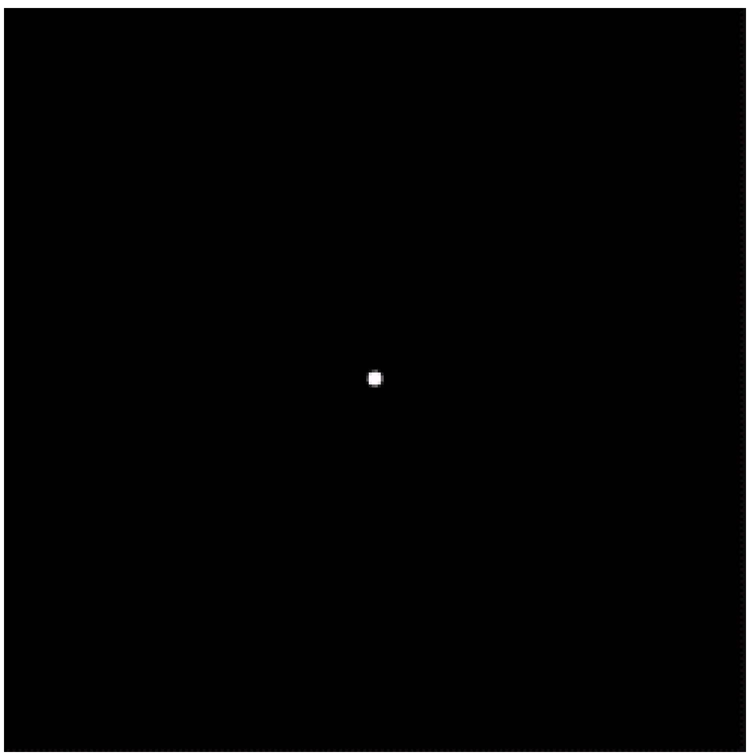
Effects of ideal low-pass filtering

- Blurring and ringing

ILPF: spatial



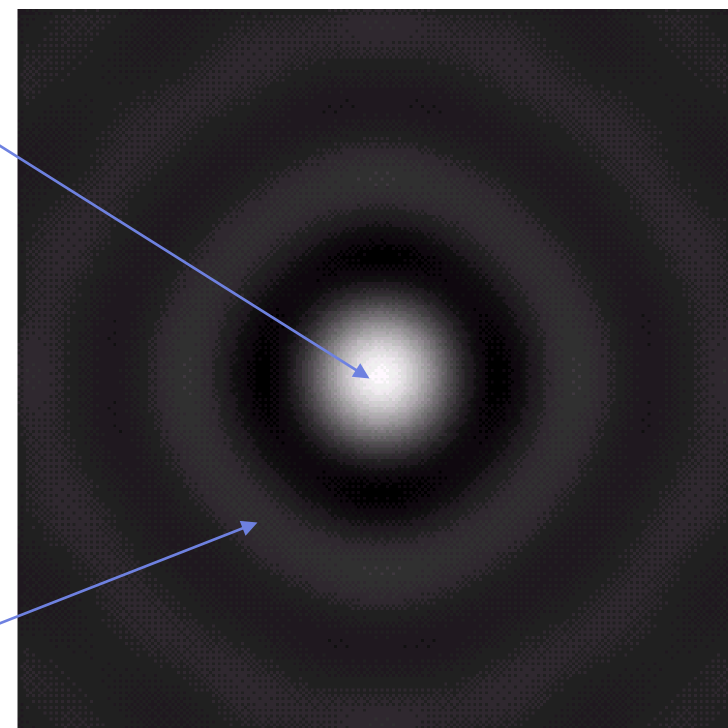
ILPF: Freq.



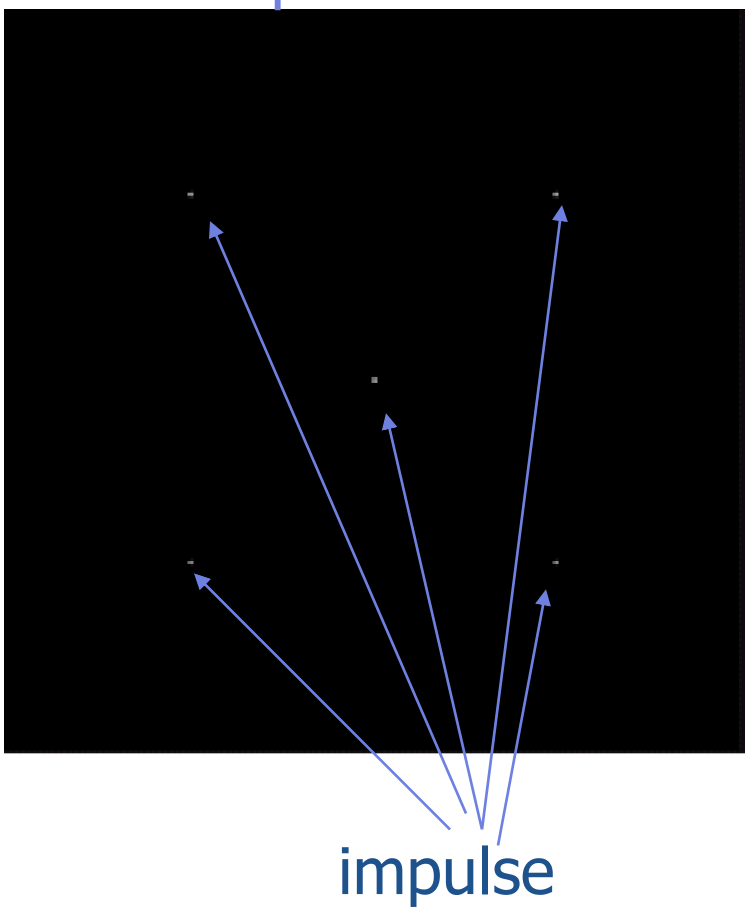
F^{-1}

blurring

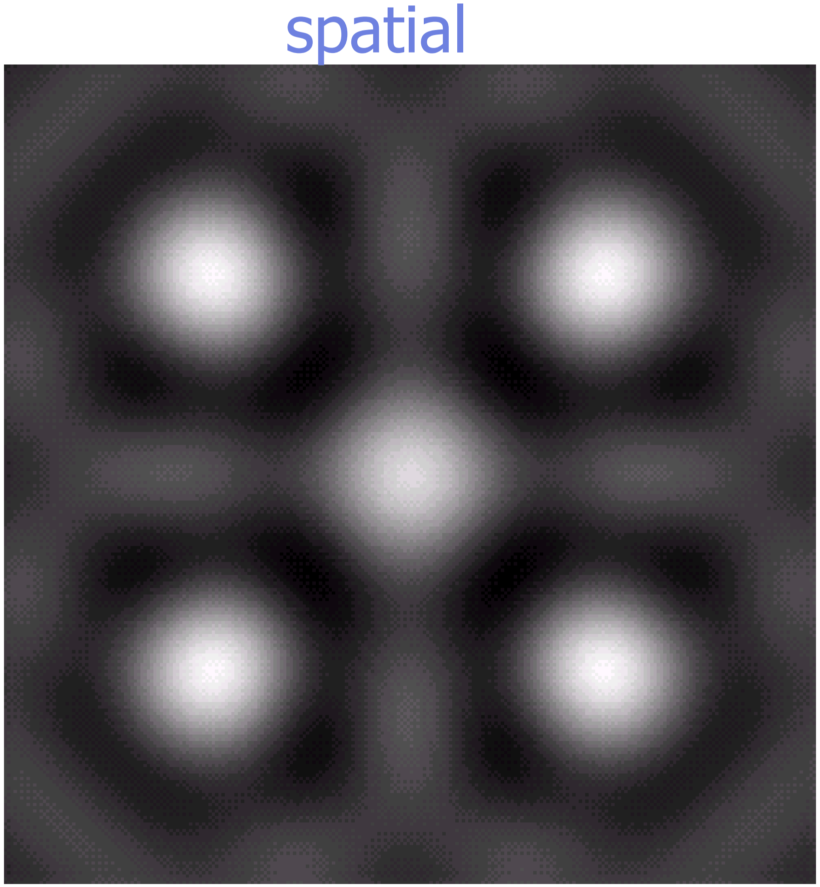
ringing



Effects of ideal low-pass filtering (cont.)



ILPF →



4.3 Smoothing Frequency-Domain Filters

- Butterworth lowpass filters (BLPF)
  - Example 4.5. (Fig. 4.15) (Fig. 4.16)
- Gussian lowpass filters (GLPFs)
  - Example 4.6. (Fig. 4.17) (Fig. 4.18) (Fig. 4.19)
- Additional Examples of Lowpass Filtering
  - Fig. 4.20
  - Fig. 4.21

Butterworth lowpass filters of order n

$$H(u,v)=\frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$

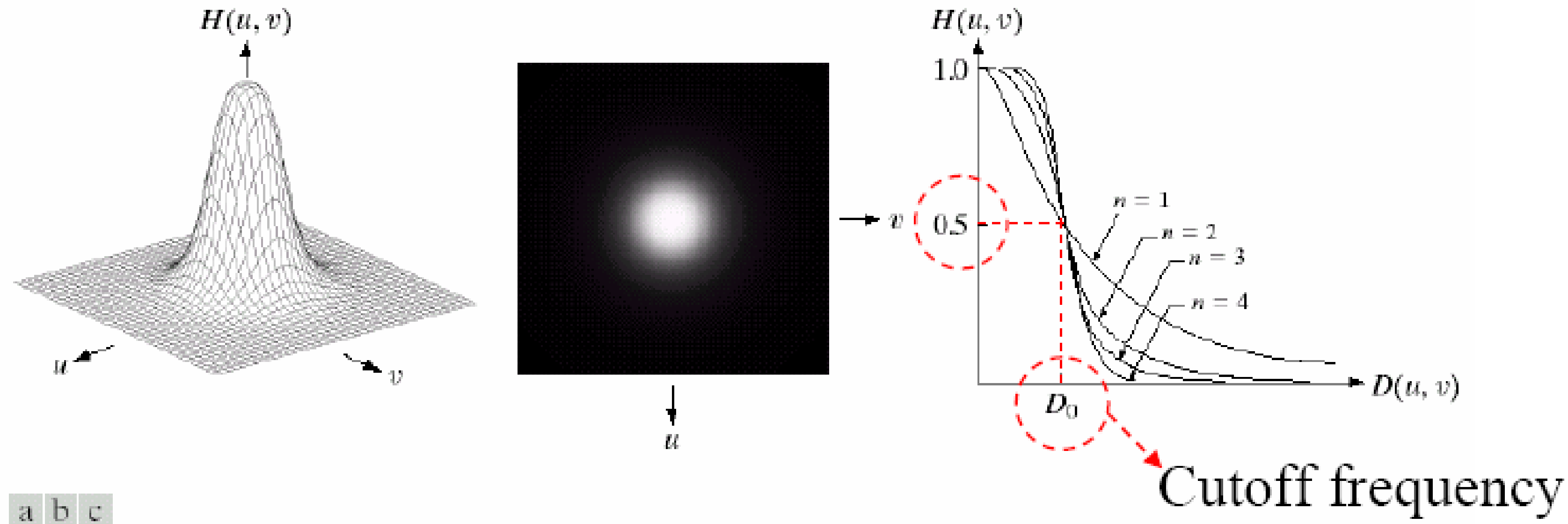
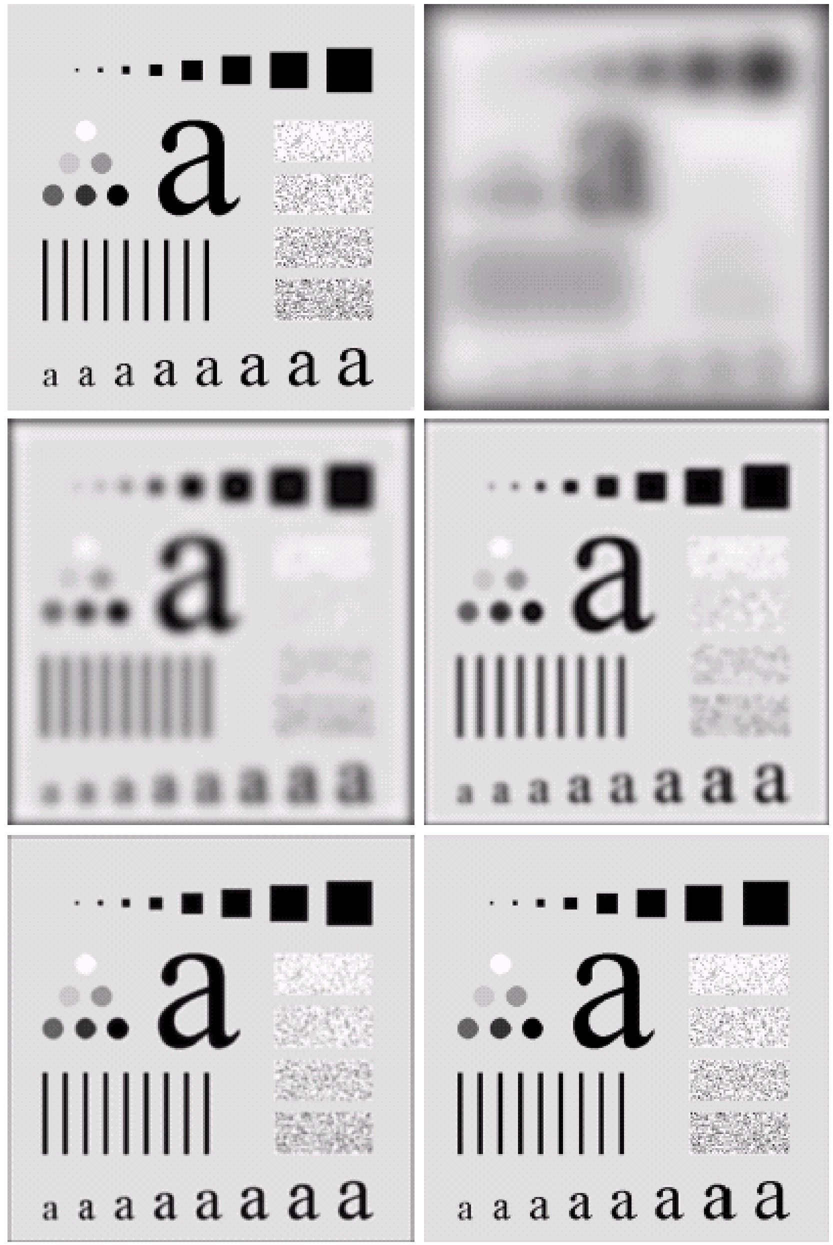


FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.



a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

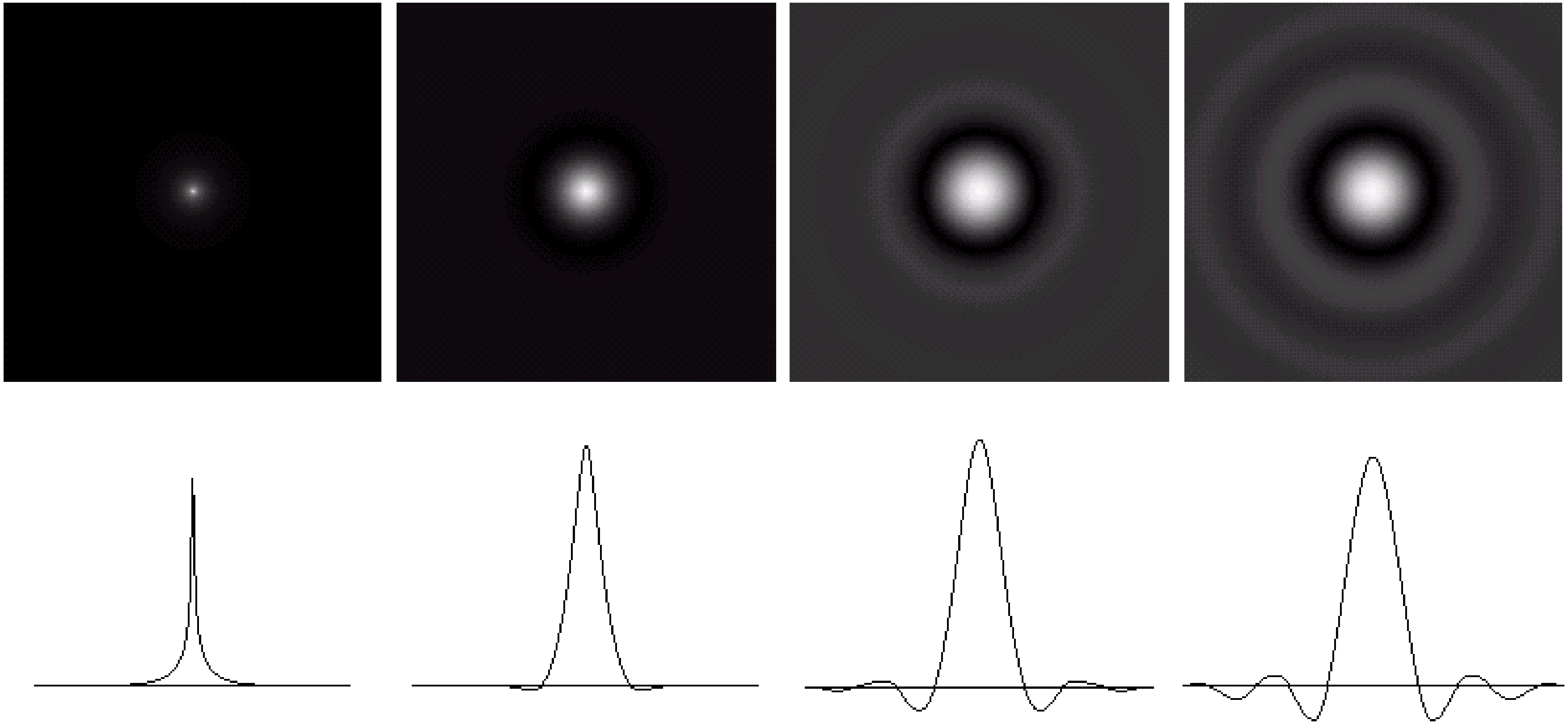


FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Gussian Lowpass filters

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

D_0 : Cutoff frequency

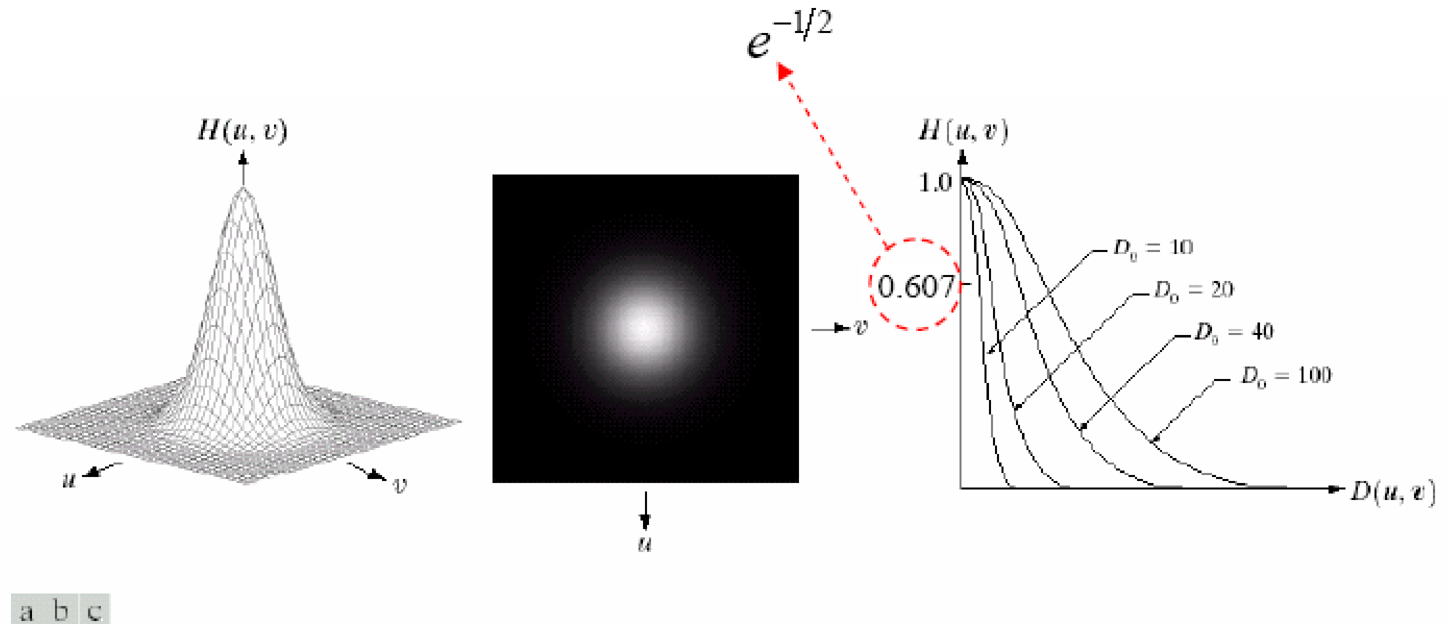


FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

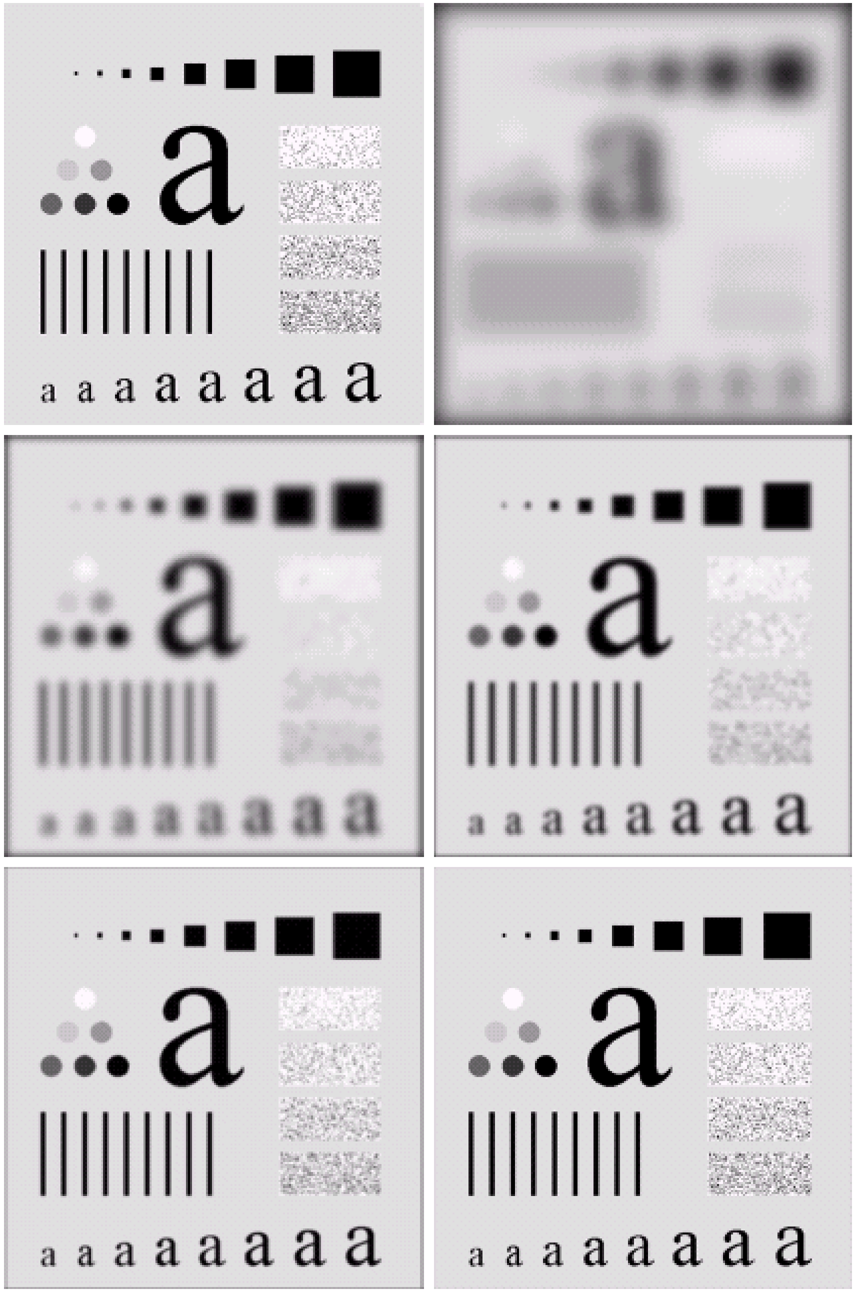


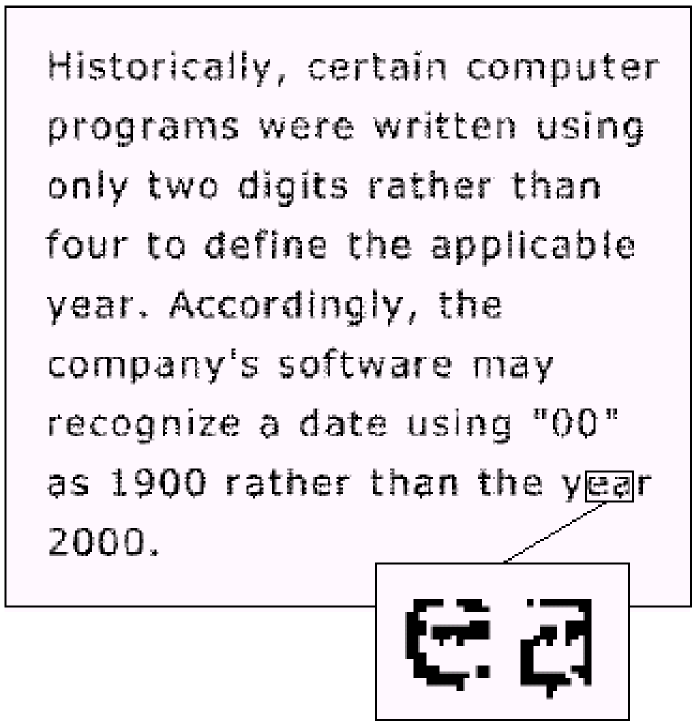
FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

a	b
c	d
e	f

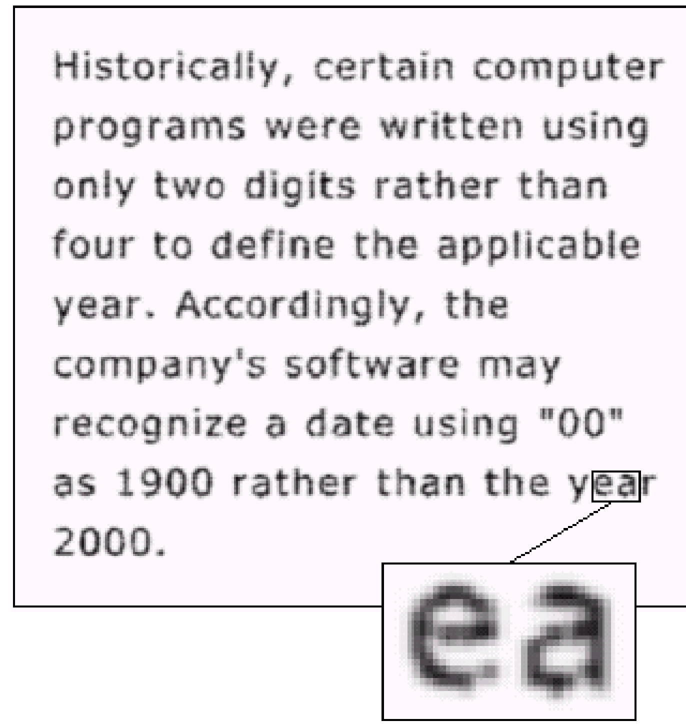
Character recognition

a b

FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view).
(b) Result of filtering with a GLPF (broken character segments were joined).



Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.



Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Reducing the effect of scan lines

a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

4.4 Sharpening Frequency-Domain Filters

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

IHPF

BHPF

GHPF

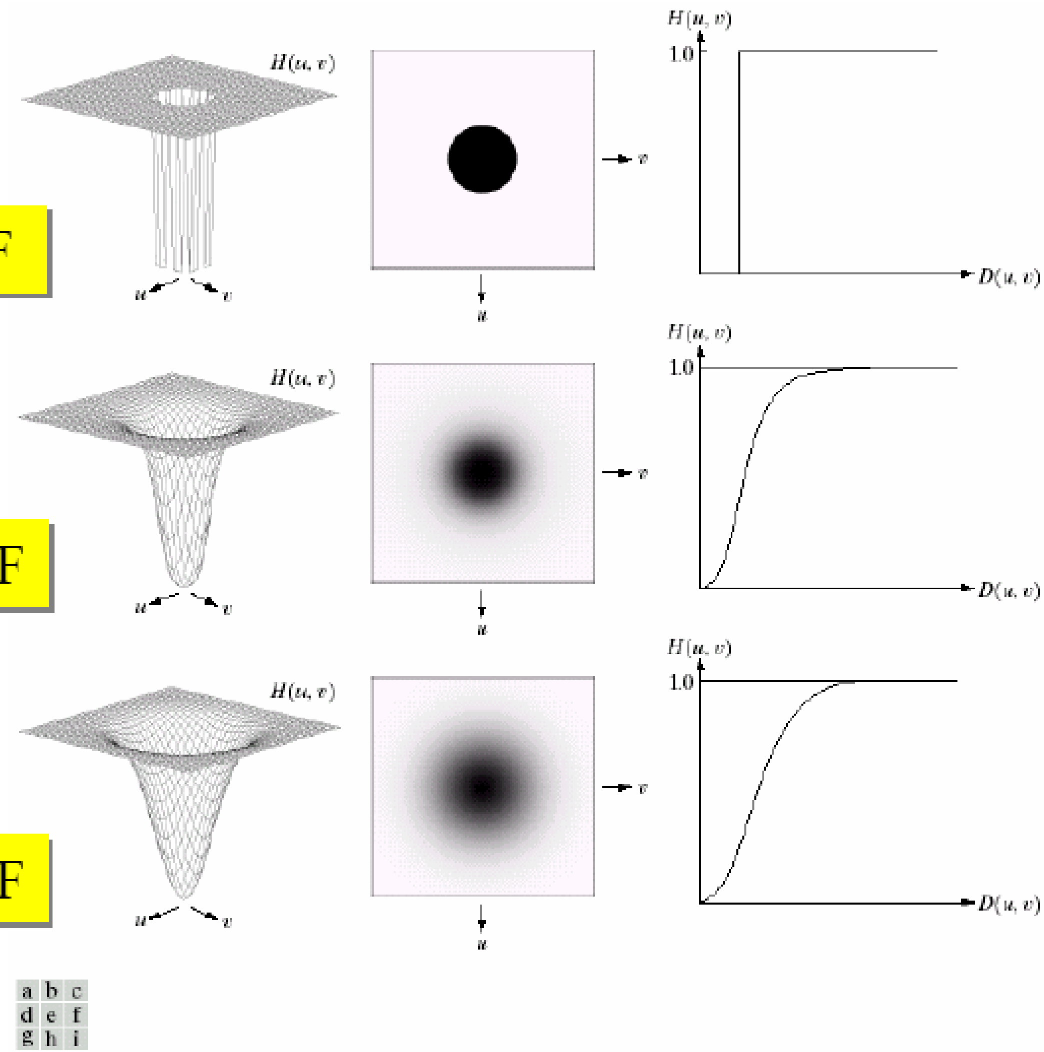
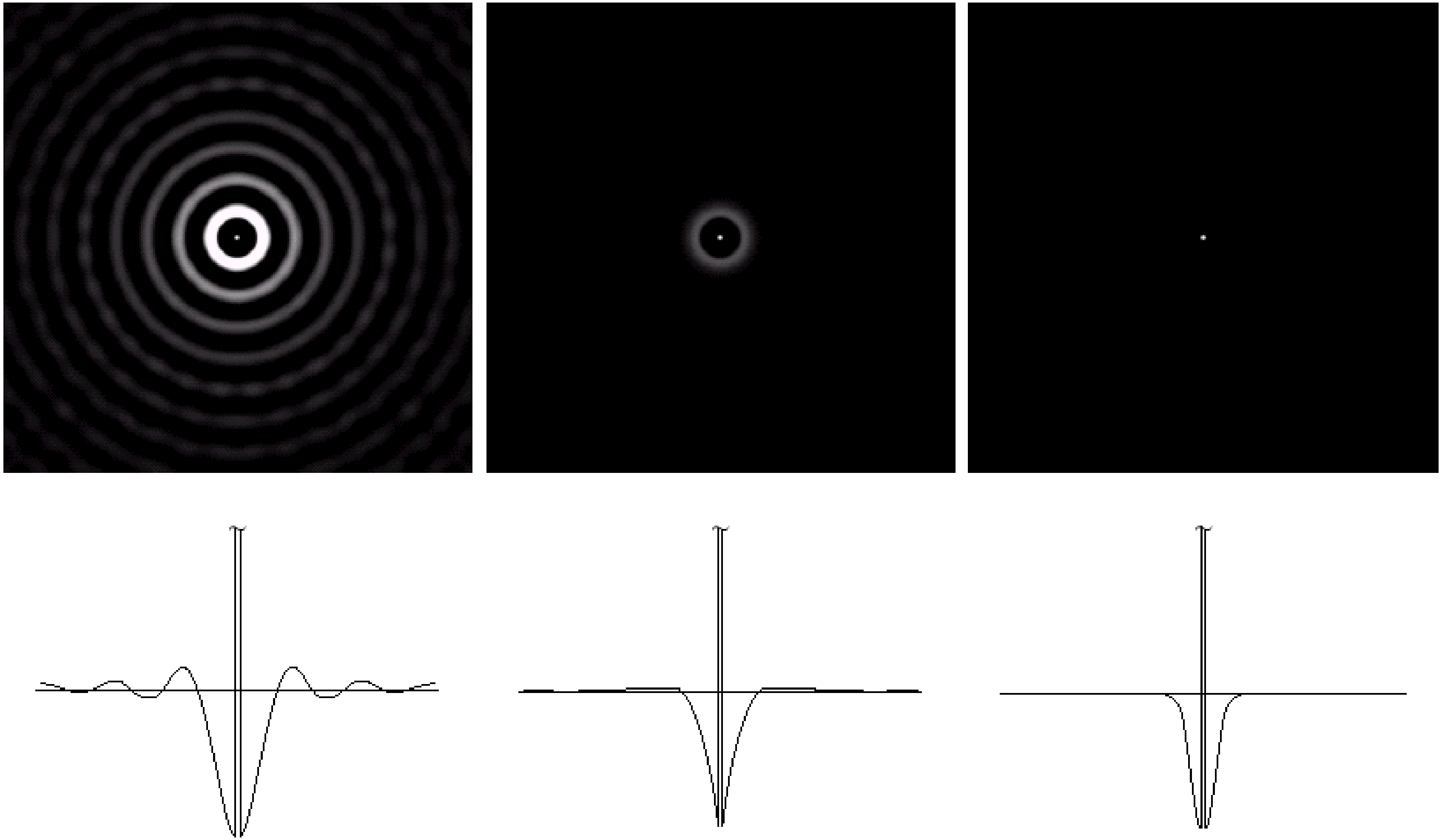


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

4.4 Sharpening Frequency-Domain Filters

- Highpass Filtering
    - Fig. 4.22
    - Fig. 4.23
  - – Ideal highpass filters
    - Fig. 4.24
  - – Butterworth highpass filters
    - Fig. 4.25
  - – Gaussian highpass filters
    - Fig. 4.26

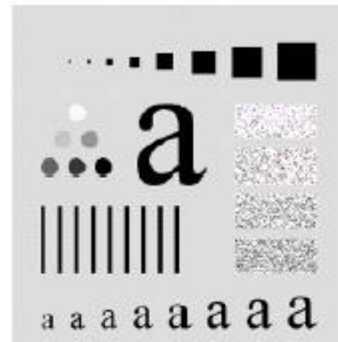


a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

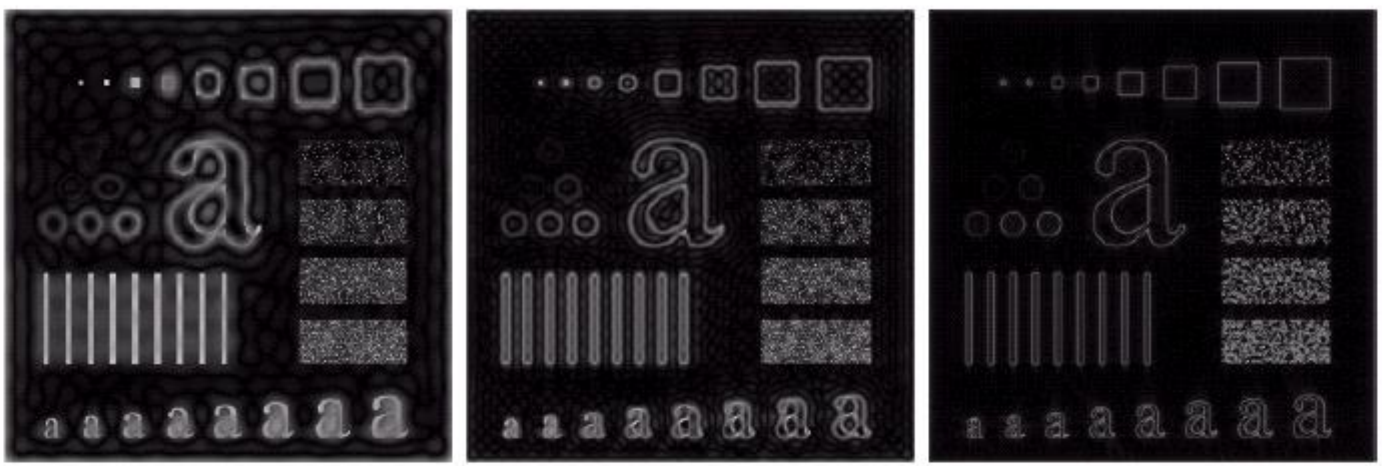
Ideal highpass filters

$$H(u,v)=\begin{cases}0 & \text{if } D(u,v)\le D_0\\ 1 & \text{if } D(u,v)>D_0\end{cases}$$



$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)>D_0\end{cases}$$

ILPF

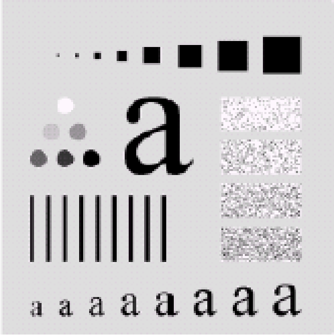


a b c

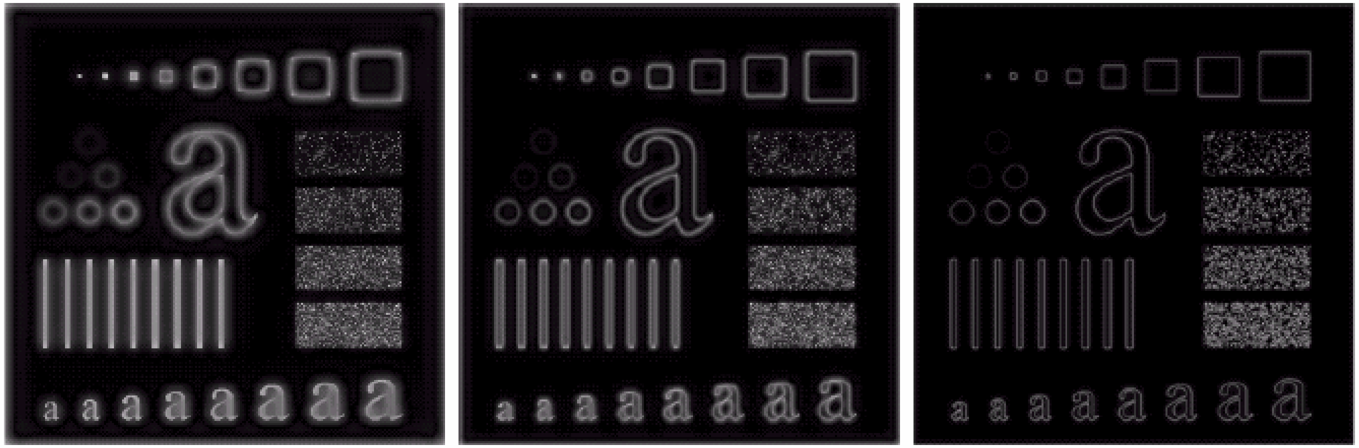
FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Butterworth highpass filters

$$H(u,v) = 1 - \frac{1}{1+\left[D(u,v)/D_0\right]^{2n}} = \frac{\left[D(u,v)/D_0\right]^{2n}}{1+\left[D(u,v)/D_0\right]^{2n}} = \frac{1}{1+\left[D_0/D(u,v)\right]^{2n}}$$



$$H(u,v) = \frac{1}{1+\left[D_0/D(u,v)\right]^{2n}}$$

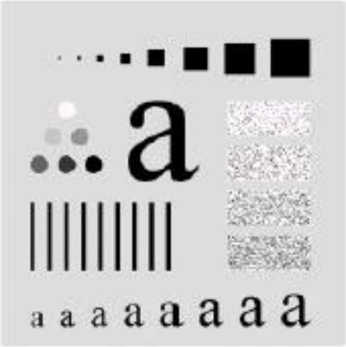


a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

Gaussian highpass filters

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

4.4 Sharpening Frequency-Domain Filters

- The Laplacian in the frequency domain
  - Fig. 4.27
  - Example 4.7: Laplacian (Fig. 4.28)
- Unsharp masking, High-boost filtering, and High-frequency emphasis filtering
  - Example 4.8: (Fig. 4.29)
  - Example 4.9: (Fig. 4.30)

Laplacian frequency-domain filters

- Spatial-domain Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

- Fourier transform

$$\Im\left[\frac{\partial^n f(x)}{\partial x^n}\right] = (ju)^n F(u)$$

$$\Im\left[\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right] = (ju)^2 F(u,v) + (jv)^2 F(u,v)$$

$$= -(u^2 + v^2) F(u,v)$$

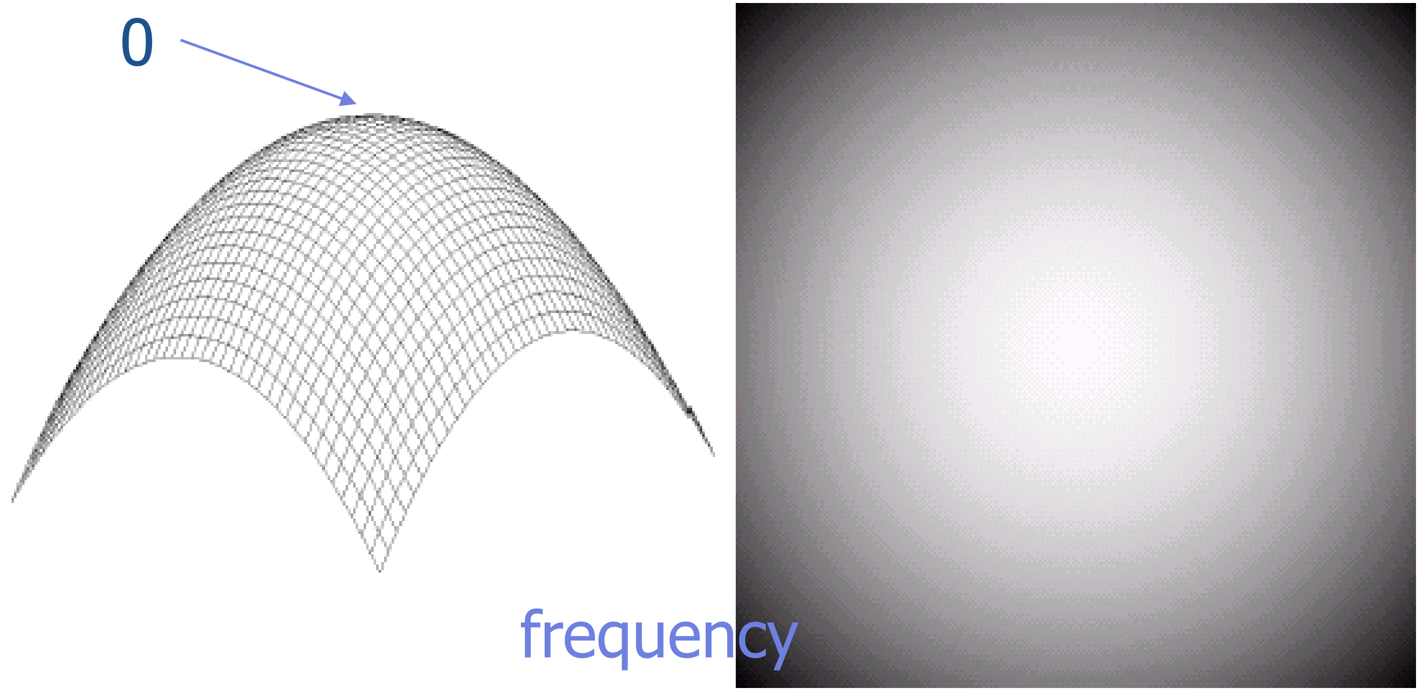
Laplacian frequency-domain filters

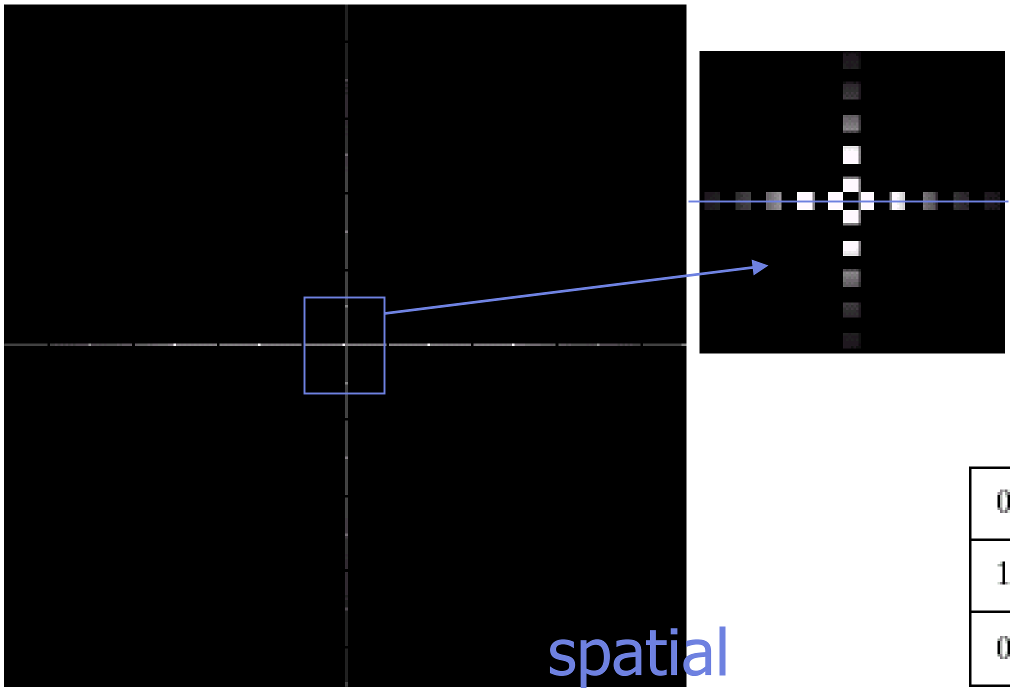
Input $f(x,y)$ → (F) → $F(u,v)$

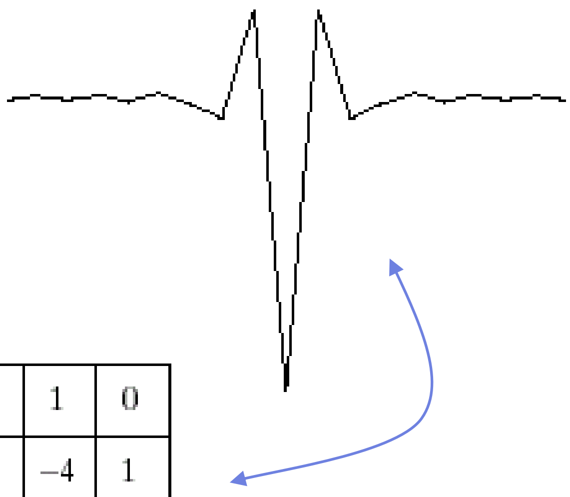
Laplacian ↓ ? $-(u^2+v^2)$ ↓

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$ → (F) → $-(u^2+v^2)F(u,v)$

The Laplacian filter in the frequency domain is
$H(u,v) = -(u^2+v^2)$



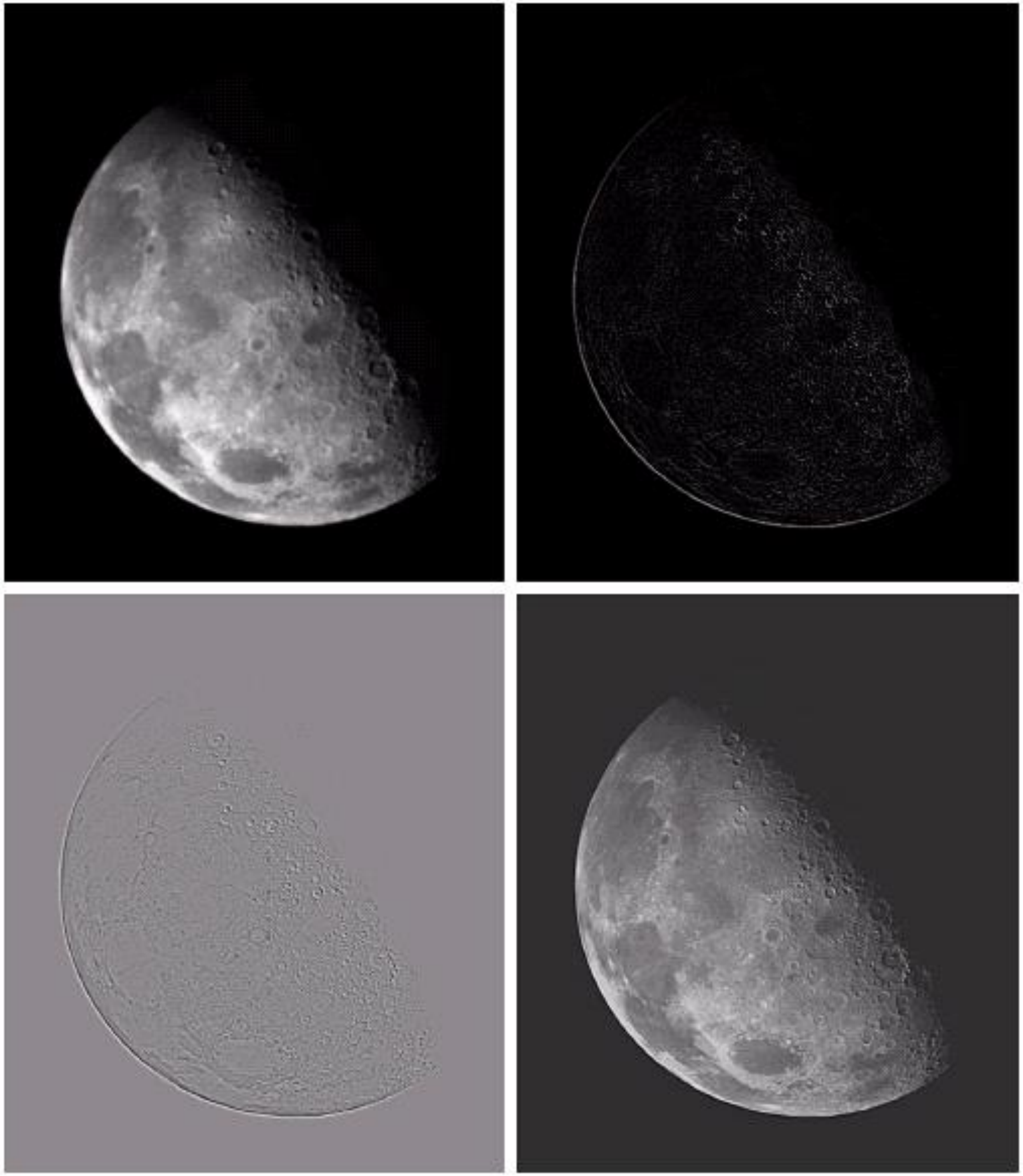




0	1	0
1	−4	1
0	1	0

original

Laplacian



Scaled Laplacian

original+ Laplacian

High-boost filtering

$$f_s(x,y) = f(x,y) - \overline{f}(x,y)$$

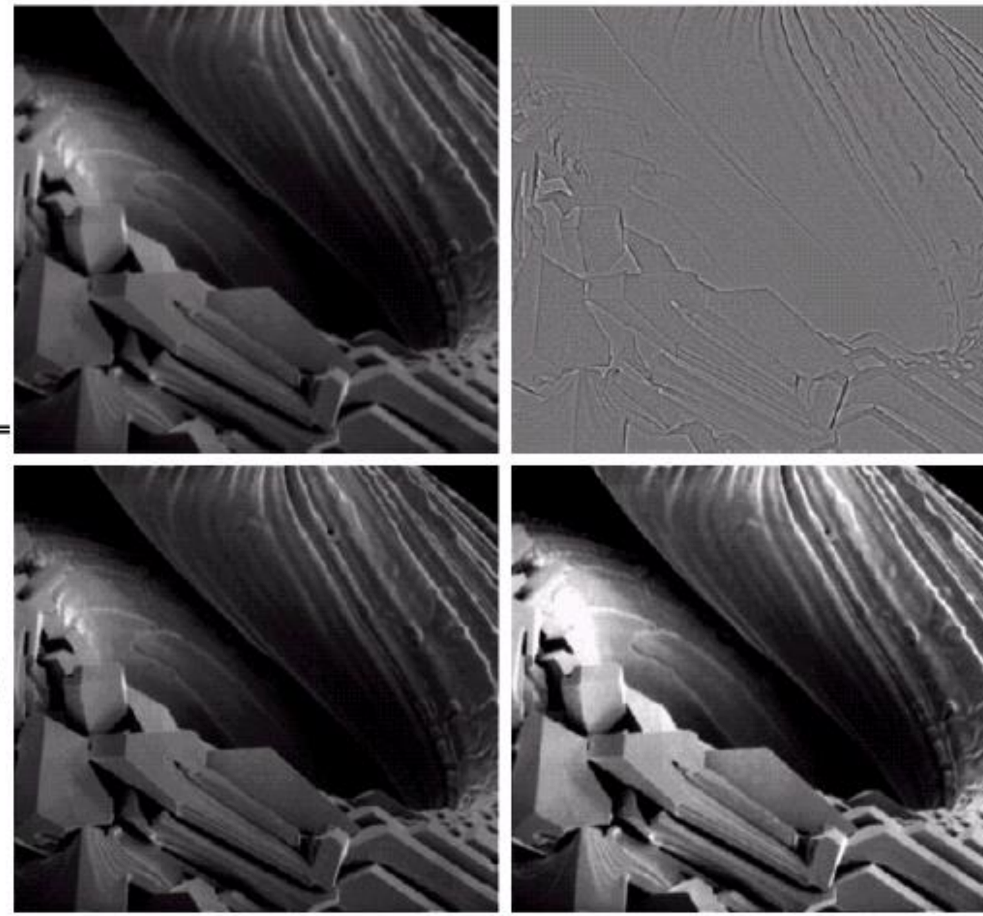
$$f_{hp}(x,y) = f(x,y) - f_{lp}(x,y)$$

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

$$f_{hb}(x,y) = Af(x,y) - f_{lp}(x,y)$$

$$\begin{aligned} f_{hb}(x,y) &= (A-1)f(x,y) + f(x,y) - f_{lp}(x,y) \\ &= (A-1)f(x,y) + f_{hp}(x,y) \end{aligned}$$

$$\boxed{H_{hb}(u,v) = (A-1) + H_{hp}(u,v)}$$



A=2 A=2.6

High-frequency emphasis filtering

$$H_{hfe}(u,v) = a + bH_{hp}(u,v)$$

$a \geq 0$ and $b > a$

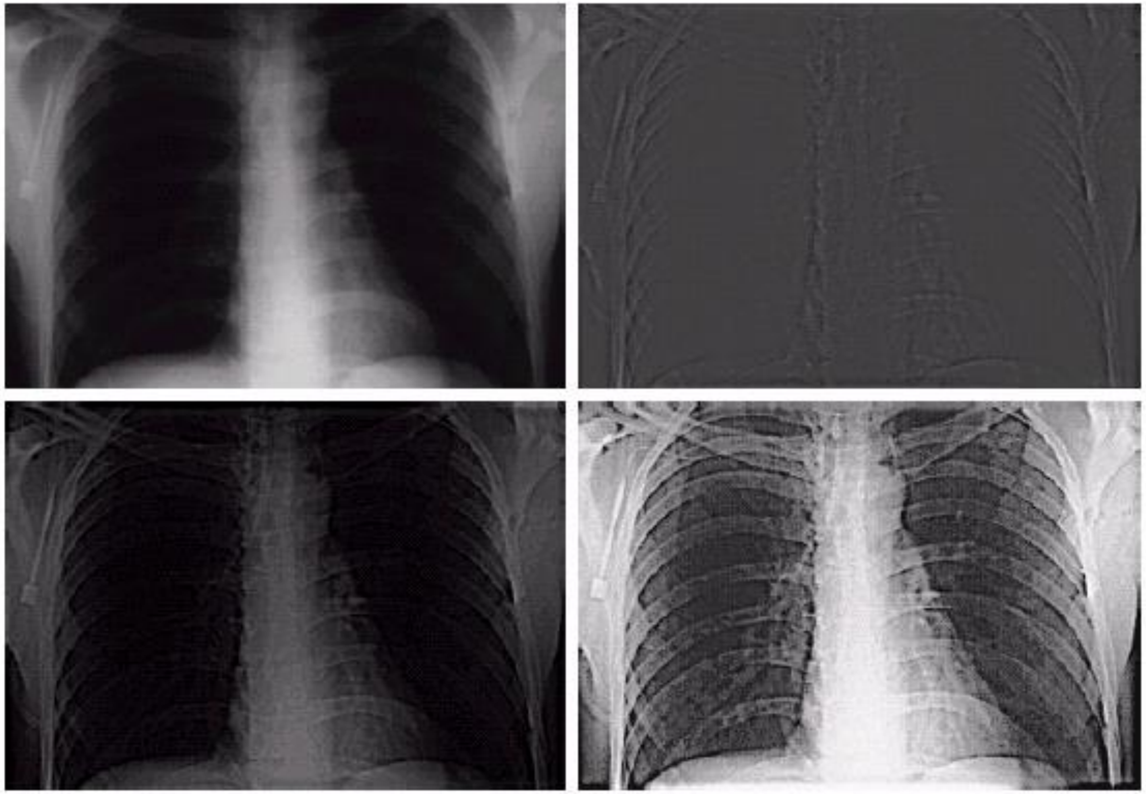


FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

$a = 0.5$
$b = 2.0$

Homomorphic filtering

- Homomorphism:
- Image formation model
  - f(x,y)=i(x,y) r(x,y)

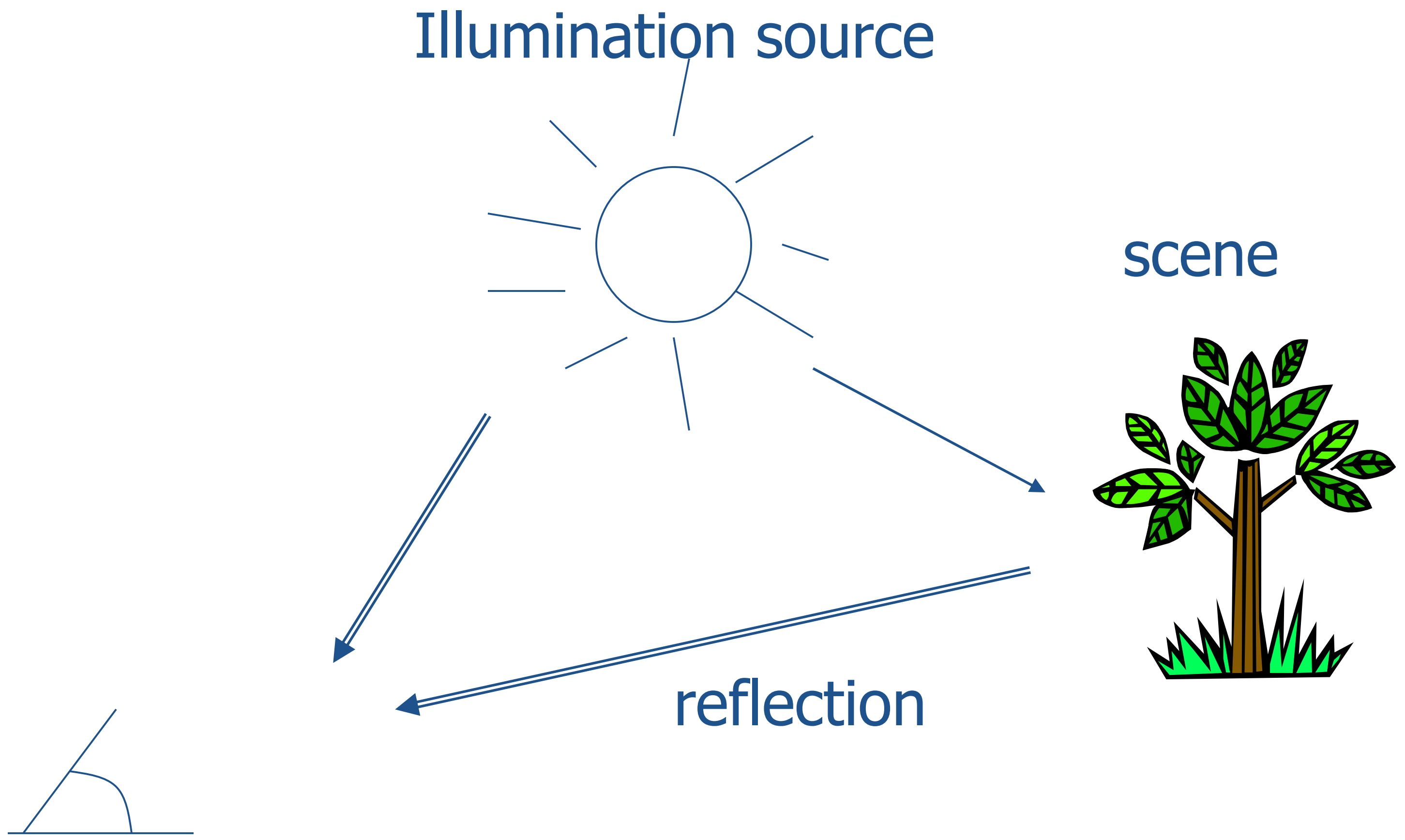
illumination:

Slow spatial variations

reflectance:

vary abruptly, particularly
at the junctions of dissimilar
objects

Image Formation Model



Homomorphic filtering

- Product term

$$\Im\left\{f(x,y)\right\}=\Im\{i(x,y)r(x,y)\}\neq\Im\{i(x,y)\}\Im\{r(x,y)\}$$

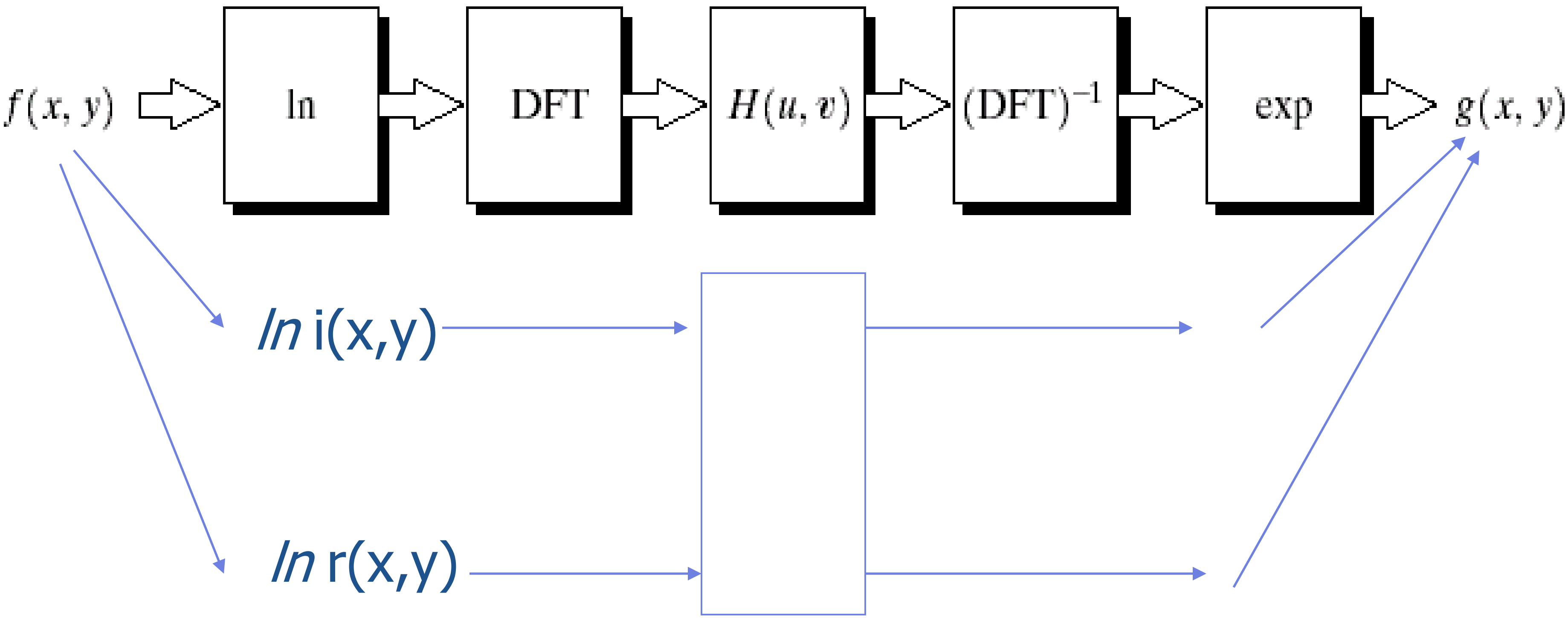
- Log of product
  - z(x,y)=*ln* f(x,y)=*ln* i(x,y)+ *ln* r(x,y)

Separation of signal source:

$$\Im\left\{z(x,y)\right\}=\Im\{\ln\ f(x,y)\}$$

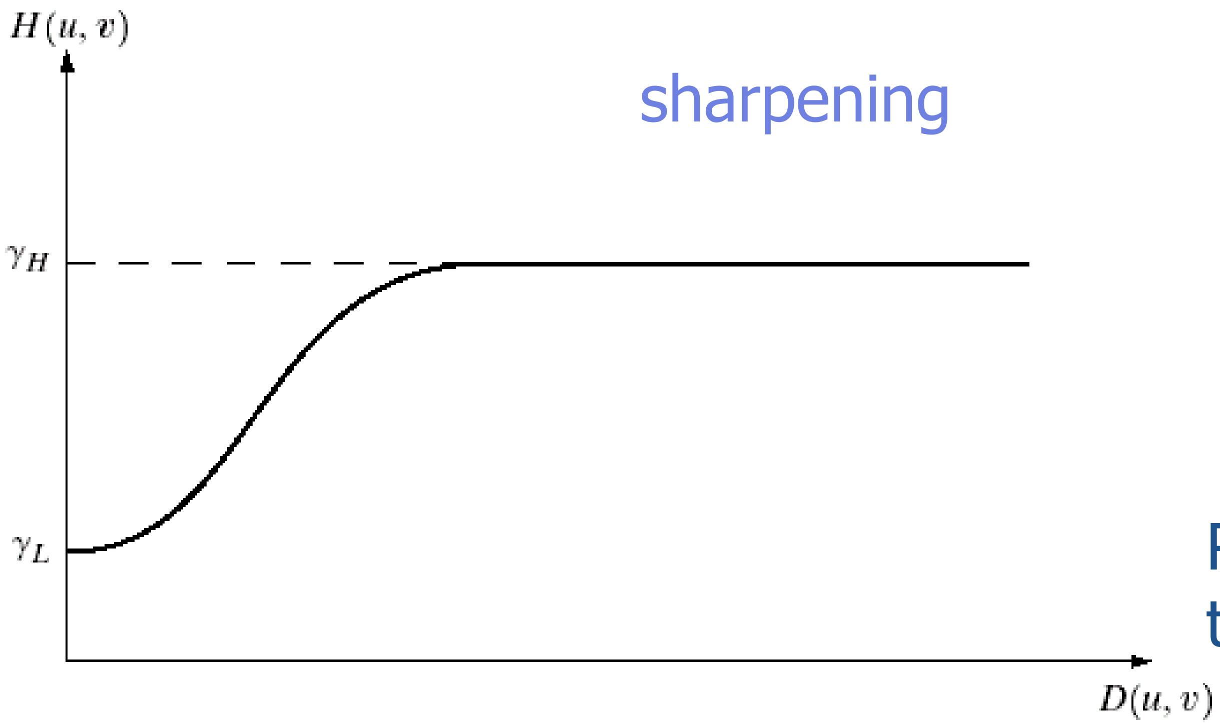
$$=\Im\{\ln i(x,y)\}+\Im\{\ln r(x,y)\}$$

Homomorphic filtering approach



How to identify the illumination and reflection

- Illumination -> low frequency
- Reflection -> high frequency



Radius from the origin

Homomophic filtering: example

original

Homomorphic filtering

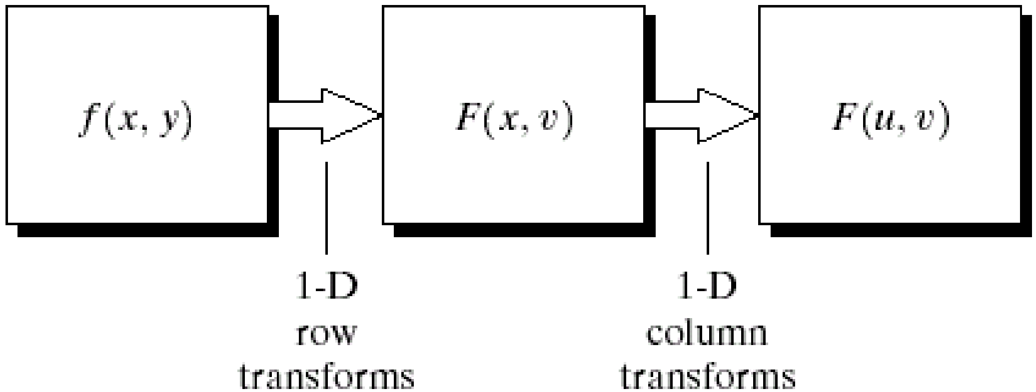


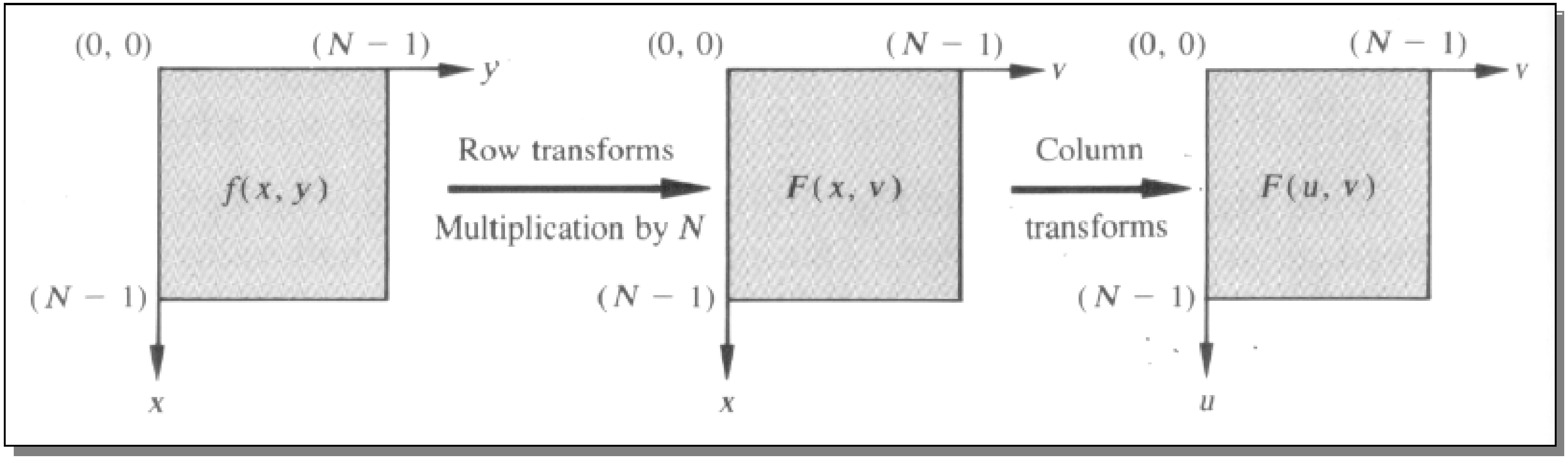
FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.

✓ The 2-D DFT Calculation with two 1-D DFT:

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)W_N^{ux}W_N^{vy}$$

$$\Rightarrow F(x,v)=\frac{1}{N}\sum_{y=0}^{N-1}f(x,y)W_N^{uy}$$

$$\Rightarrow F(u,v)=N\left(\frac{1}{N}\sum_{x=0}^{N-1}F(x,v)W_N^{ux}\right)$$



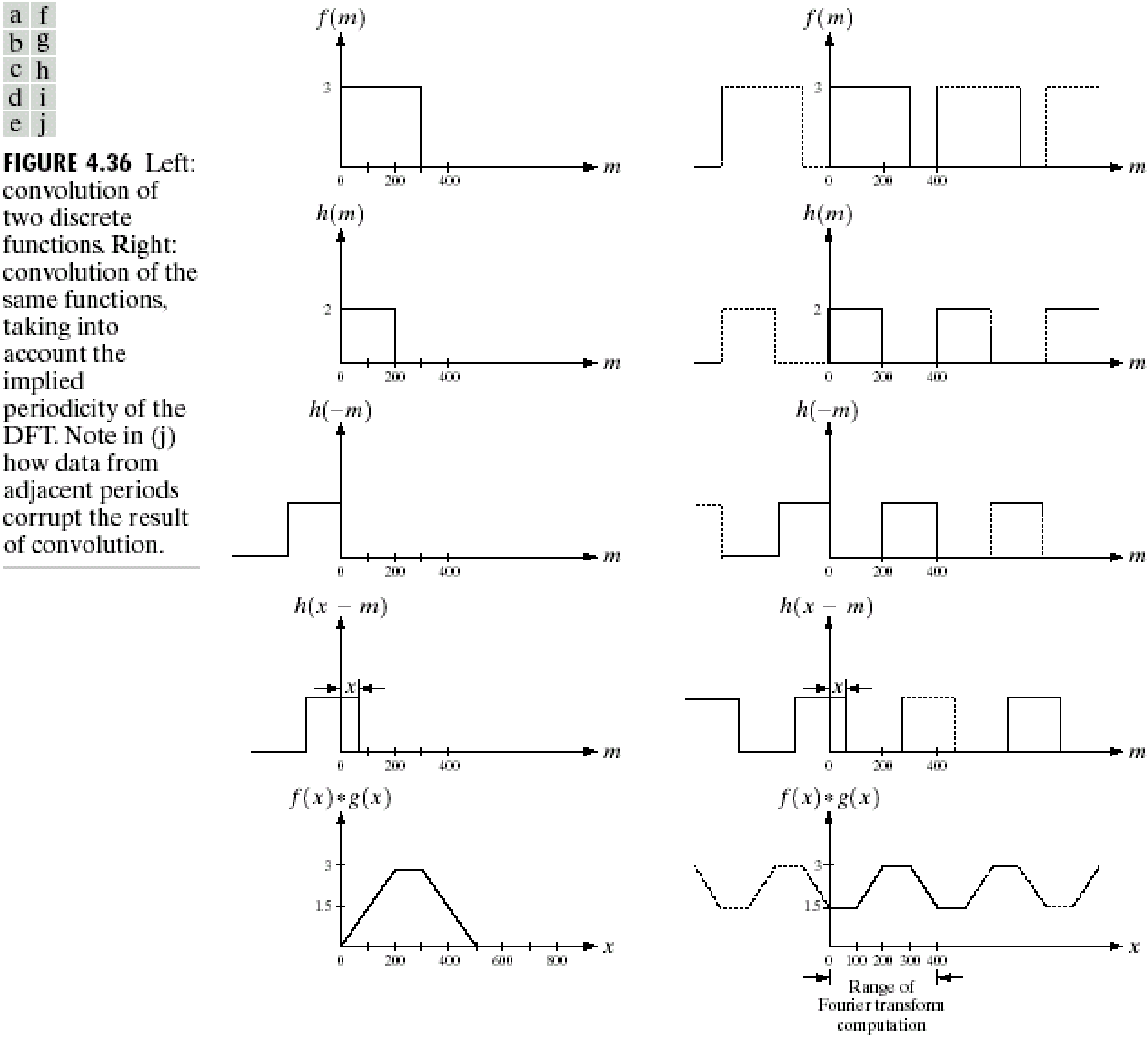
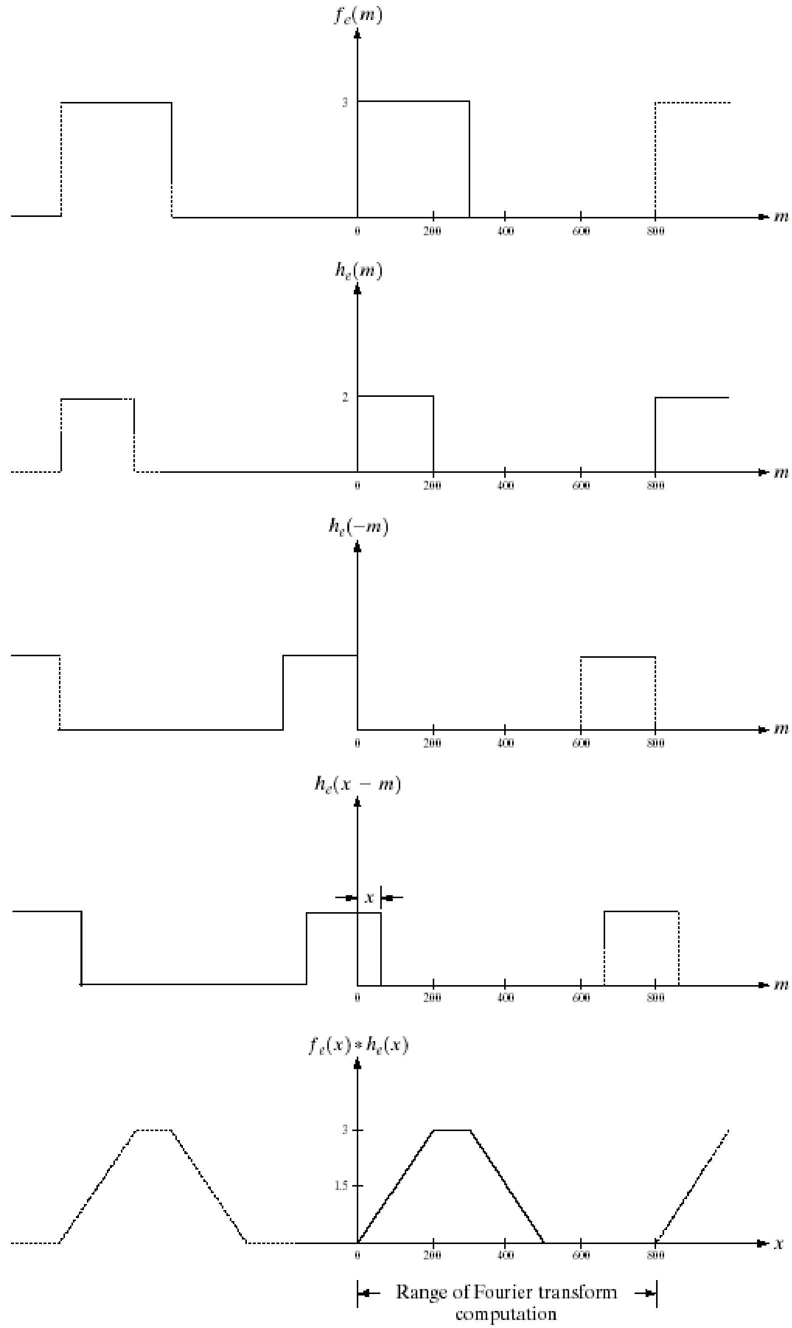


FIGURE 4.36 Left: convolution of two discrete functions. Right: convolution of the same functions, taking into account the implied periodicity of the DFT. Note in (j) how data from adjacent periods corrupt the result of convolution.

a
b
c
d
e

FIGURE 4.37 Result of performing convolution with extended functions. Compare Figs. 4.37(e) and 4.36(e).



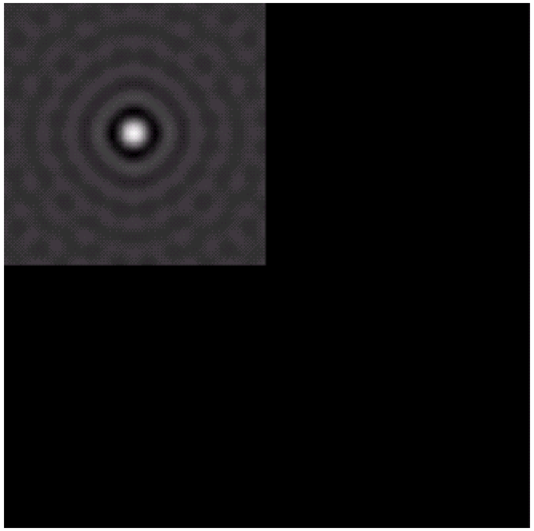
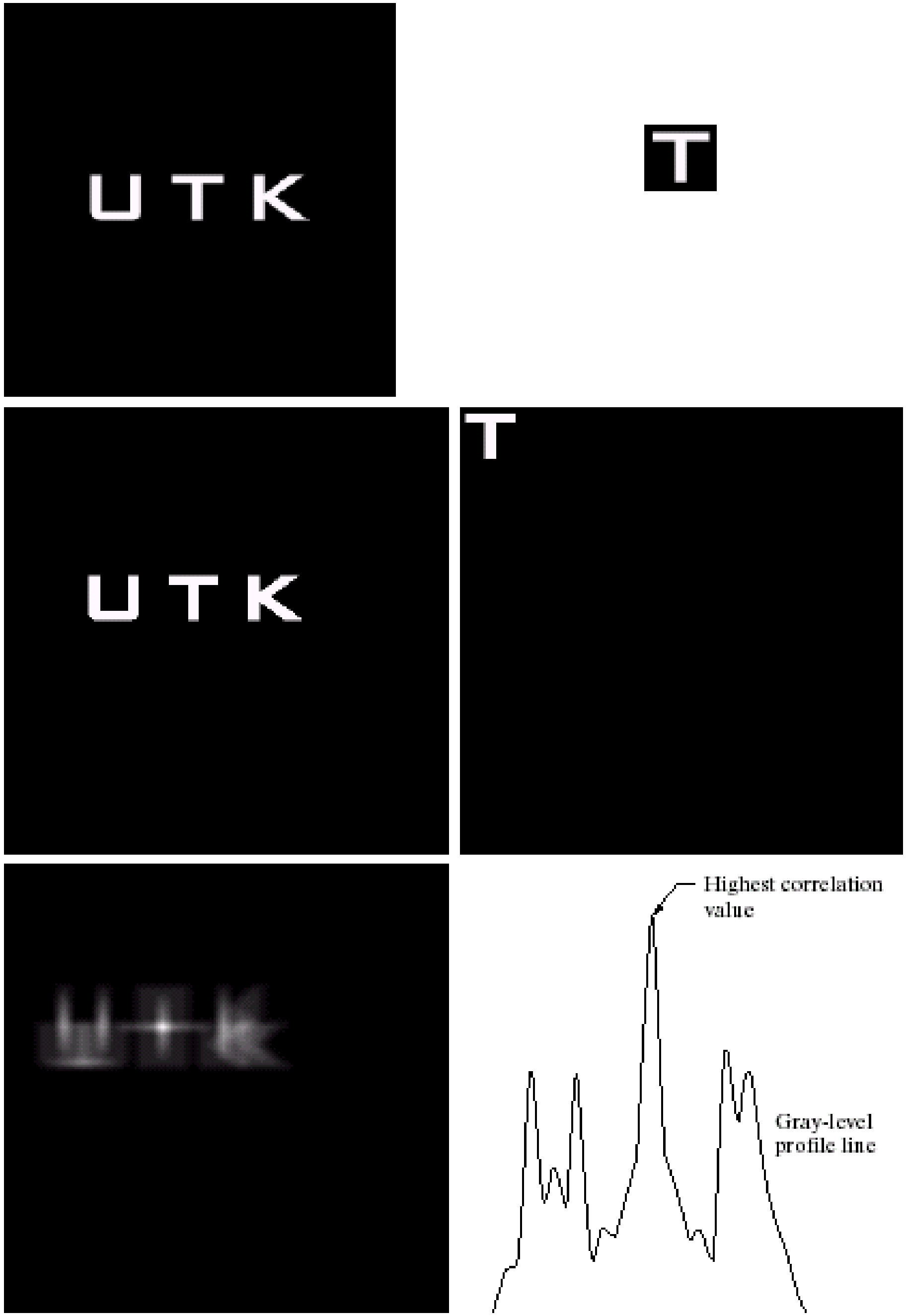


FIGURE 4.39 Padded lowpass filter is the spatial domain (only the real part is shown).

FIGURE 4.40 Result of filtering with padding. The image is usually cropped to its original size since there is little valuable information past the image boundaries.



a b
c d
e f

FIGURE 4.41
(a) Image.
(b) Template.
(c) and
(d) Padded images.
(e) Correlation function displayed as an image.
(f) Horizontal profile line through the highest value in (e), showing the point at which the best match took place.

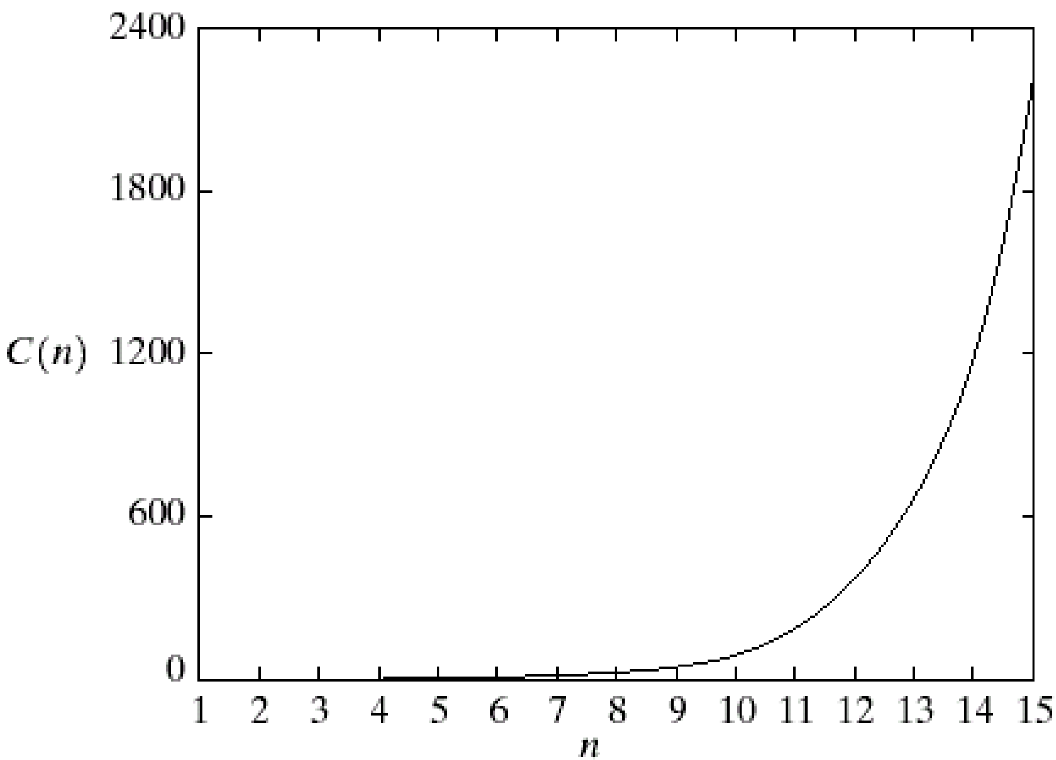


FIGURE 4.42 Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of n.